APPM 1350 **Final Exam** **Summer 2020**

On the front page please write your name and clearly label each problem This exam is worth 100 points and has 4 questions on both sides of this paper.

- Make sure all of your work is on separate sheets of paper. Nothing on this exam sheet will be graded. Please begin each problem on a new page.
- **Show all work and simplify your answers!** Name any theorem that you use. Answers with no justification will receive no points unless the problem explicitly states otherwise.
- Notes, papers, calculators, cell phones, and other electronic devices are not permitted, except for a computer for proctoring through Zoom.
- You must use methods that we have learned in class thus far to solve the problems. (Dominance of powers is not valid)

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1. (35 pts) Limits and Continuity (The following questions are unrelated)

   (a) Find the following limits:

      i. $\lim\limits_{t\to 0} \frac{\sqrt{1-t}-\sqrt{1+t}}{t}$

      ii. $\lim\limits_{x\to\infty} e^{-x}\sin(x)$

      iii. $\lim\limits_{\theta\to 0} \frac{\sin(\theta^3)}{\sin^3(2\theta)}$

   (b) Does $f(x) = x^3 - \cos(2x) + \sin(x)$ cross the x-axis on the interval $[0, \pi]$? Explain.

   Solution:

   (a) i.

$$\begin{aligned}
\lim_{t\to 0} \frac{\sqrt{1-t}-\sqrt{1+t}}{t} &= \lim_{t\to 0} \frac{\sqrt{1-t}-\sqrt{1+t}}{t}\left(\frac{\sqrt{1-t}+\sqrt{1+t}}{\sqrt{1-t}+\sqrt{1+t}}\right) \\
&= \lim_{t\to 0} \frac{(1-t)-(1+t)}{t(\sqrt{1-t}+\sqrt{1+t})} \\
&= \lim_{t\to 0} \frac{-2t}{t(\sqrt{1-t}+\sqrt{1+t})} \\
&= \lim_{t\to 0} \frac{-2}{\sqrt{1-t}+\sqrt{1+t}} = \frac{-2}{\sqrt{1-0}+\sqrt{1+0}} = -1
\end{aligned}$$

      ii. First, note that $-1 \leq \sin(x) \leq 1$. Then, $-e^{-x} \leq e^{-x}\sin(x) \leq e^{-x}$ and taking the limit of the upper and lower bound, $\lim\limits_{x\to\infty} -e^{-x} = \lim\limits_{x\to\infty} e^{-x} = 0$. And therefore, by the Squeeze Theorem, $\lim\limits_{x\to\infty} e^{-x}\sin(x) = 0$.

iii. Using the fact that $\lim_{u\to 0} \frac{\sin(u)}{u} = 1$, where u is an arbitrary function of θ,

$$\begin{aligned}
\lim_{\theta\to 0} \frac{\sin(\theta^3)}{\sin^3(2\theta)} &= \lim_{\theta\to 0} \frac{\sin(\theta^3)}{\sin^3(2\theta)} \frac{(2\theta)^3}{(2\theta)^3} = \lim_{\theta\to 0} \frac{\sin(\theta^3)}{8\theta^3} \frac{(2\theta)^3}{\sin^3(2\theta)} \\
&= \frac{1}{8} \lim_{\theta\to 0} \frac{\sin(\theta^3)}{\theta^3} \lim_{\theta\to 0} \left(\frac{2\theta}{\sin(2\theta)}\right)^3 = \frac{1}{8} \lim_{\theta\to 0} \frac{\sin(\theta^3)}{\theta^3} \left(\lim_{\theta\to 0} \frac{2\theta}{\sin(2\theta)}\right)^3 \\
&= \frac{1}{8}(1)(1)^3 = \frac{1}{8}
\end{aligned}$$

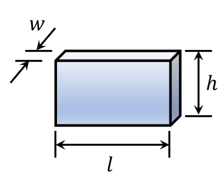
Alternatively, using L'Hopital's rule,

$$\begin{aligned}
\lim_{\theta\to 0} \frac{\sin(\theta^3)}{\sin^3(2\theta)} &= \frac{0}{0} \\
&\overset{L'H}{=} \lim_{\theta\to 0} \frac{3\theta^2\cos(\theta^3)}{3\sin^2(2\theta)(2\cos(\theta))} = \lim_{\theta\to 0} \frac{\theta^2\cos(\theta^3)}{2\sin^2(2\theta)\cos(\theta)} \\
&= \underbrace{\lim_{\theta\to 0} \frac{\cos(\theta^3)}{\cos(\theta)}}_{=1} \lim_{\theta\to 0} \frac{\theta^2}{2\sin^2(2\theta)} = \frac{0}{0} \\
&\overset{L'H}{=} \lim_{\theta\to 0} \frac{2\theta}{4\sin(2\theta)(2\cos(\theta))} = \frac{1}{8} \lim_{\theta\to 0} \frac{2\theta}{\sin(2\theta)} \underbrace{\lim_{\theta\to 0} \frac{1}{\cos(2\theta)}}_{=1} = \frac{0}{0} \\
&\overset{L'H}{=} \frac{1}{8} \lim_{\theta\to 0} \frac{2}{2\cos(2\theta)} = \frac{1}{8}(1) = \frac{1}{8}
\end{aligned}$$

(b) First, note that $f(x)$ is continuous because polynomials, $\sin(x)$, and $\cos(x)$ are continuous everywhere. Next, note that $f(0) = (0)^3 - \cos(0) + \sin(0) = -1 < 0$ and that $f(\pi) = (\pi)^3 - \cos(2\pi) + \sin(\pi) = \pi^3 - 1 > 0$. And by the Intermediate Value Theorem, the function must cross the x-axis at least once.

2. (25 pts) Derivatives (The following questions are unrelated)

(a) Suppose that you are flying with an airline that requires that the sum of the dimensions of carry-on luggage $(l + w + h)$ must be no more than 90 cm and the length must be twice the height (see illustration).



What is the largest volume that such a carry-on can hold and the corresponding dimensions?

(b) $\frac{e^{x/y}}{xy} = 1$. Find $\frac{dy}{dx}$.

Solution:

(a) Note that we want to maximize the volume of the suitcase or $V = lwh = (2h)wh = 2wh^2$. The luggage is constrained that $l+w+h = 2h+w+h = 3h+w = 90$. Therefore, $w = 90-3h$ and the function for the volume is $V(h) = 2(90-3h)h^2 = 180h^2 - 6h^3$. $V'(h) = 360h - 18h^2 = 18h(20-h)$. $V''(h) = 360 - 36h$ and $V''(0) = 360$ and $V''(20) = -360$ so $h = 20$ cm is a maximum. From the constraints, $l = 2h = 40$ and $w = 90 - 3(20) = 30$. So, the dimensions of the luggage with maximum volume are $40 \times 30 \times 20$ cm and the volume is 24000 cm^3.

(b) Taking the natural log of both sides,

$$\ln\left(\frac{e^{x/y}}{xy}\right) = \ln(1) = 0$$
$$\frac{x}{y} - \ln(x) - \ln(y) = 0$$

Taking the derivative of both sides,

$$\frac{d}{dx}\left[\frac{x}{y} - \ln(x) - \ln(y)\right] = 0$$
$$\frac{y - x\frac{dy}{dx}}{y^2} - \frac{1}{x} - \frac{1}{y}\frac{dy}{dx} = 0$$
$$\frac{1}{y} - \frac{1}{x} = \frac{x}{y^2}\frac{dy}{dx} + \frac{1}{y}\frac{dy}{dx} = \left(\frac{x}{y^2} + \frac{1}{y}\right)\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{\frac{1}{y} - \frac{1}{x}}{\frac{x}{y^2} + \frac{1}{y}} = \frac{xy - y^2}{x^2 + xy}$$

Alternatively, we can use quotient, product, and chain rule:

$$\frac{d}{dx}\left[\frac{e^{x/y}}{xy} = 1\right]$$
$$0 = \frac{xy(e^{x/y})(y - xy')/y^2 - e^{x/y}(xy' + y)}{(xy)^2}$$
$$= \frac{x(e^{x/y})(y - xy')/y - e^{x/y}(xy' + y)}{x^2y^2}$$
$$= e^{x/y}\left[\frac{1}{xy^2} - \frac{1}{y^3}y' - \frac{1}{xy^2}y' - \frac{1}{x^2y}\right]$$
$$\frac{1}{y^3}y' + \frac{1}{xy^2}y' = \left[\frac{1}{y^3} + \frac{1}{xy^2}\right]y' = \frac{1}{xy^2} - \frac{1}{x^2y}$$
$$y' = \frac{\frac{1}{xy^2} - \frac{1}{x^2y}}{\frac{1}{y^3} + \frac{1}{xy^2}} = \frac{\frac{x-y}{x^2y^2}}{\frac{x+y}{xy^3}}$$
$$= \frac{(x-y)y}{(x+y)x} = \frac{xy - y^2}{x^2 + xy}$$

TURN OVER—More problems on the back!

3. (20 pts) Integrals (The following questions are unrelated)

   (a) Integrate

$$\int_{\ln(2)}^{\ln(3)} \frac{e^{2x}}{e^{2x}-1}\,dx$$

   (b) Let the function $g(x)$ be defined as

$$g(x) = \int_{\sqrt{x}}^{1} \cos(t^2)dt$$

   on the interval $(0, 2\pi)$. What are the intervals of increase and decrease? Justify.

Solution:

   (a) Let $u = e^{2x} - 1$ and $du = 2e^{2x}\,dx$. Then $\frac{du}{2} = e^{2x}\,dx$ and the lower bound becomes $e^{2\ln(2)} - 1 = e^{\ln(4)} - 1 = 4 - 1 = 3$ and the upper bound becomes $e^{2\ln(3)} - 1 = e^{\ln(9)} - 1 = 9 - 1 = 8$. Then substituting and integrating,

$$\frac{1}{2}\int_{3}^{8} \frac{1}{u}\,du = \frac{1}{2}\ln|u|\Big|_{3}^{8} = \frac{1}{2}[\ln(8) - \ln(3)] = \frac{1}{2}\ln(8/3) = \ln\left(2\sqrt{\frac{2}{3}}\right)$$

   (b)

$$g(x) = -\int_{1}^{\sqrt{x}} \cos(t^2)dt$$

   and from the Fundamental Theorem of Calculus Part I,

$$g'(x) = -\cos((\sqrt{x})^2)\frac{d}{dx}\left[\sqrt{x}\right] = -\frac{\cos(x)}{2\sqrt{x}}$$

   Note that on $(0, 2\pi)$, $\sqrt{x} > 0$, and thus the sign of the derivative only depends on the sign of $\cos(x)$. And so the interval of increase is $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and the interval of decrease is $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$

4. (20 pts) Inverses, Logs, and Exponentials (The following questions are unrelated)

   (a) Suppose $f(x) = \frac{1}{1+x^2}$ on the interval $(-\infty, 0]$.

      i. Find $f^{-1}(x)$.

      ii. What is the domain and range of $f^{-1}(x)$?

      iii. Find $(f^{-1})'(1/2)$ without finding the derivative of f^{-1}.

   (b) Polonium-210 has a half-life of 138 days. How long would it take for a quantity of Polonium to decay to 1/10 of its original amount?

Solution:

   (a) i. Let $y = \frac{1}{1+x^2}$. Then

$$\begin{aligned} 1 + x^2 &= \frac{1}{y} \\ x^2 &= \frac{1}{y} - 1 \\ x &= \pm\sqrt{\frac{1}{y} - 1} \end{aligned}$$

and based on the given domain of the function, $x = -\sqrt{\frac{1}{y} - 1}$ and therefore,

$$f^{-1}(x) = -\sqrt{\frac{1}{x} - 1}$$

ii. The domain of $f(x)$ is $(-\infty, 0]$ and the range of $f(x)$ is $(0, 1]$. Therefore, the domain of $f^{-1}(x)$ is $(0, 1]$ and the range of $f^{-1}(x)$ is $(-\infty, 0]$.

iii. We use the equation that $(f^{-1})'(1/2) = \frac{1}{f'(f^{-1}(1/2))}$. First,

$$f'(x) = -\frac{2x}{(1+x^2)^2}$$

Solving for the x-value at which the function equals 1/2,

$$\begin{aligned} \frac{1}{2} &= \frac{1}{1+x^2} \\ 1 + x^2 &= \frac{1}{1/2} = 2 \\ x &= \pm 1 \end{aligned}$$

However, only $x = -1$ is in the domain of the function. Therefore,

$$(f^{-1})'(1/2) = \frac{1}{f'(-1)} = \frac{1}{-2(-1)/(1+1^2)^2} = 2$$

(b) The form of the equation for the mass as a function of time is

$$m(t) = m_0 e^{kt}$$

Solving for k,

$$\begin{aligned} \frac{m_0}{2} &= m_0 e^{k(138)} \\ \ln(1/2) = -\ln(2) &= 138k \\ k &= -\frac{\ln(2)}{138} \end{aligned}$$

Then $m(t) = m_0 e^{-\frac{\ln(2)}{138}t}$ and solving for when $m(t) = \frac{m_0}{10}$,

$$\begin{aligned} \frac{m_0}{10} &= m_0 e^{-\frac{\ln(2)}{138}t} \\ \ln(1/10) = -\ln(10) &= -\frac{\ln(2)}{138}t \end{aligned}$$

And so $t = \frac{138\ln(10)}{\ln(2)}$ days. (Approximately 458.4 days)