APPM 1350 **Exam 3** **Summer 2020**

On the front page please write your name and clearly label each problem This exam is worth 100 points and has 4 questions on both sides of this paper.

- Make sure all of your work is on separate sheets of paper. Nothing on this exam sheet will be graded. Please begin each problem on a new page.
- **Show all work and simplify your answers!** Name any theorem that you use. Answers with no justification will receive no points unless the problem explicitly states otherwise.
- Notes, papers, calculators, cell phones, and other electronic devices are not permitted, except for a computer for proctoring through Zoom.
- You must use methods that we have learned in class thus far to solve the problems. (Dominance of powers is not valid)

---

1. (20 pts) Unrelated, short answer questions.

   (a) Given $x_1 = 1$ and $f(x) = -x^3 - x^2 + 4$, find x_2 using Newton's method.

   (b) Find f (generalized anti-derivative):

      i. $f'(x) = x^{2/3} + \frac{1}{x^3} + \frac{1}{\sqrt{x}}$

      ii. $f''(\theta) = \theta + \sin(\theta) - \cos(\theta)$

   Solution:

   (a) Using Newton's method, $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$. $f'(x) = -3x^2 - 2x$, so $x_2 = 1 - \left(\frac{2}{-5}\right) = \frac{7}{5}$.

   (b) Anti-differentiating,

      i. $f(x) = \frac{x^{5/3}}{5/3} + \frac{x^{-2}}{-2} + 2x^{1/2} + C = \frac{3x^{5/3}}{5} - \frac{1}{2x^2} + 2\sqrt{x} + C$.

      ii. $f'(\theta) = \frac{\theta^2}{2} - \cos(\theta) - \sin(\theta) + C$ and
      $f(\theta) = \frac{\theta^3}{6} - \sin(\theta) + \cos(\theta) + C\theta + D$ where C, D are arbitrary constants.

2. (13 pts) A person standing on the edge of a cliff throws a ball upward at 10 m/s. The ball hits the ground below at 90 m/s. Assume that the ball experiences a constant acceleration of 10 m/s^2 downward. How tall is the cliff? Use anti-differentiation.

   Solution: It is given that $a(t) = -10$ m/s^2. Anti-differentiating, $v(t) = -10t + C$. Note that $v(0) = -10(0) + C = C = 10$ m/s from the problem statement, so $C = 10$. We are also given that the ball hits the ground at 90 m/s, so we can use the velocity equation to solve for the time at which the ball hits the ground: $-10t^* + 10 = -90$, so $-10t^* = -100$ and $t^* = 10$ s. Anti-differentiating again, $s(t) = -5t^2 + 10t + D$. If we pick the reference frame where the ground is at 0, $s(10) = -5(10)^2 + 10(10) + D = -500 + 100 + D = -400 + D = 0$ and so $D = 400$. And therefore,

   $$s(t) = -5t^2 + 10t + 400$$

   and $s(0)$ (when the ball is released) is 400 m and so the cliff is 400 meters tall.

3. (22 pts) Let $f(x) = x^2 + x$ on the interval $[-1, 0]$.

(a) Set up the Riemann sum for this function and the given interval. Use n equally spaced subintervals and right endpoints.

(b) Simplify the sum. (i.e. find a value that no longer has summations)

(c) Using limit rules, evaluate the limit of the simplified sum as $n \to \infty$.

(d) Compute using a definite integral.

Solution:

(a) $\Delta x = \frac{b-a}{n} = \frac{0-(-1)}{n} = \frac{1}{n}$ and $x_i = a + i\Delta x = -1 + \frac{i}{n}$. Then the Riemann sum becomes

$$\sum_{i=1}^{n} f(x_i)\Delta x = \frac{1}{n}\sum_{i=1}^{n}\left[\left(-1+\frac{i}{n}\right)^2 + \left(-1+\frac{i}{n}\right)\right]$$

(b) Simplifying the sum,

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}\left[\left(-1+\frac{i}{n}\right)^2 + \left(-1+\frac{i}{n}\right)\right] &= \frac{1}{n}\sum_{i=1}^{n}\left[\left(1-2\frac{i}{n}+\frac{i^2}{n^2}\right) + \left(-1+\frac{i}{n}\right)\right] \\
&= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{i^2}{n^2}-\frac{i}{n}\right] = \frac{1}{n^3}\sum_{i=1}^{n} i^2 - \frac{1}{n^2}\sum_{i=1}^{n} i \\
&= \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} - \frac{1}{n^2}\frac{n(n+1)}{2} \\
&= \frac{2n^2+3n+1}{6n^2} - \frac{n+1}{2n}
\end{aligned}$$

(c) Taking the limit of this expression as $n \to \infty$,

$$\lim_{n\to\infty} \frac{2n^2+3n+1}{6n^2} - \frac{n+1}{2n} = \lim_{n\to\infty} \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} - \frac{1}{2} + \frac{1}{2n} = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$$

(d) Alternatively,

$$\int_{-1}^{0} x^2 + x dx = \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{0} = 0 - \left(\frac{1}{2} - \frac{1}{3}\right) = -\frac{1}{6}$$

which is what we found using Riemann sums.

4. (12 pts) Using the Fundamental Theorem of Calculus, find $f'(x)$ if

$$f(x) = \int_{\cos(x)}^{x^3} \frac{\sin(t)}{t} dt$$

Solution: By FTC-1,

$$f'(x) = \frac{\sin(x^3)}{x^3}\frac{d}{dx}[x^3] - \frac{\sin(\cos(x))}{\cos(x)}\frac{d}{dx}[\cos(x)] = \frac{3\sin(x^3)}{x} + \tan(x)\sin(\cos(x))$$

TURN OVER—More problems on the back!

5. (33 pts) Find the following:

(a)

$$\int \frac{x}{\sqrt{x^2+1}}dx$$

(b)

$$\int_0^{\pi/3} \frac{\sin(\theta)}{\cos^3(\theta)}\, d\theta$$

(c) The average value of $f(t) = \frac{1}{t^2} + t$ on $[1, 3]$.

Solution:

(a) Let $u = x^2 + 1$. Then $du = 2x\, dx$ and the integral becomes

$$\frac{1}{2}\int \frac{1}{\sqrt{u}}\, du$$

and integrating and substituing the definition of u,

$$\frac{1}{2}\int \frac{1}{\sqrt{u}}du = \frac{1}{2}\frac{u^{1/2}}{1/2} + C = \sqrt{u} + C = \sqrt{x^2+1} + C$$

(b) Let $u = \cos(\theta)$. Then $du = -\sin(\theta)\, d\theta$ and transforming the bounds, the lower bound becomes $\cos(0) = 1$ and the upper bound becomes $\cos(\pi/3) = \frac{1}{2}$. Then the integral becomes

$$-\int_1^{1/2} \frac{1}{u^3}\, du$$

and integrating,

$$-\int_1^{1/2} \frac{1}{u^3}\, du = \int_{1/2}^{1} u^{-3}\, du = -\frac{1}{2u^2}\bigg|_{1/2}^{1} = -\frac{1}{2} + 2 = \frac{3}{2}$$

(c)

$$\begin{aligned} f_{avg} &= \frac{1}{3-1}\int_1^3 \frac{1}{t^2} + t\, dt = \frac{1}{2}\int_1^3 t^{-2} + t\, dt \\ &= \frac{1}{2}\left[-\frac{1}{t} + \frac{t^2}{2}\right]_1^3 = \frac{1}{2}[(-1/3 + 9/2) - (-1 + 1/2)] \\ &= \frac{1}{2}[5 - 1/3] = \frac{1}{2}\left(\frac{14}{3}\right) = \frac{7}{3} \end{aligned}$$

Formulas: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ $\quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ $\quad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$