
On the front page please write your name and clearly label each problem This exam is worth 100 points and has 4 questions on both sides of this paper.

- Make sure all of your work is on separate sheets of paper. Nothing on this exam sheet will be graded. Please begin each problem on a new page.
- **Show all work and simplify your answers!** Name any theorem that you use. Answers with no justification will receive no points unless the problem explicitly states otherwise.
- Notes, papers, calculators, cell phones, and other electronic devices are not permitted, except for a computer for proctoring through Zoom.
- You must use methods that we have learned in class thus far to solve the problems.

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1. (10 pts) Short answer. If true, state why and if false, give a counterexample.

   (a) If the error in the radius of a circle is ± 0.1 m and the radius is 10 m, then the approximate error in its area is 2π m^2.

   (b) A value can be a critical point or a point of inflection but not both.

   Solution:

   (a) This is true. The governing equation is $A = \pi r^2$ and so $\frac{dA}{dr} = 2\pi r$ and so $dA = 2\pi r dr$ and so $dA = 2\pi(10)(0.1) = 2\pi$ in^2.

   (b) This is false. Consider $f(x) = x^3$. $f'(x) = 3x^2$ which has a critical point at $x = 0$ and $f''(x) = 6x$ which has a point of inflection at $x = 0$. Therefore, $x = 0$ is both a critical point and a point of inflection.

2. (24 pts) Find $\frac{dy}{dx}$ for the following equations.

   (a) $y = (x^2 + \sin(x^2))^7$

   (b) $y = \frac{\sqrt{4-x^2}}{x^2}$

   (c) $\cos(xy) + x^2y^2 = 1$

   Solution:

   (a)

$$\begin{aligned}\frac{dy}{dx} &= 7[x^2 + \sin(x^2)]^6 \frac{d}{dx}[x^2 + \sin(x^2)] \\ &= 7[x^2 + \sin(x^2)]^6(2x + \cos(x^2)(2x)) \\ &= 14x[x^2 + \sin(x^2)]^6(1 + \cos(x^2))\end{aligned}$$

   (b)

$$\frac{dy}{dx} = \frac{(x^2)\frac{1}{2\sqrt{4-x^2}}(-2x) - \sqrt{4-x^2}(2x)}{(x^2)^2} = \frac{-\frac{x^3}{2\sqrt{4-x^2}} - 2x\sqrt{4-x^2}}{x^4}$$

(c)

$$\begin{aligned}
\frac{d}{dx}[\cos(xy) + x^2y^2 = 1] \\
\sin(xy)[y + x\frac{dy}{dx}] + 2xy^2 + 2x^2y\frac{dy}{dx} &= 0 \\
(x\sin(xy) + 2x^2y)\frac{dy}{dx} &= -y\sin(xy) - 2xy^2 \\
\frac{dy}{dx} &= -\frac{y\sin(xy) + 2xy^2}{x\sin(xy) + 2x^2y}
\end{aligned}$$

3. (10 pts) You're burning a log (idealized as a cylinder) on the fire. The fire evenly consumes the log such that the radius decreases by 3 inches every hour. Supposing that the log initially has a radius of 6 inches, how fast is the cross-sectional area of the log decreasing after the log has been on the fire for 40 minutes? Show all work.

Solution: We are given that $\frac{dr}{dt} = -3$ in/hr. We want to find $\frac{dA}{dt}\Big|_{t=40min} = \frac{dA}{dt}\Big|_{r=4}$. The governing equation is $A = \pi r^2$. Implicitly differentiating both sides, $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$ and so

$$\frac{dA}{dt}\Big|_{r=4in} = 2\pi(4)(-3) = -24\pi \text{ in}^2/\text{hr}$$

4. (24 pts) The following questions are not related. Justify your answers and cite any theorems you use.

(a) Approximate $\sqrt{8}$ using linearization.

(b) Find the absolute maximum and minimum of $f(x) = x^3 + \frac{3}{2}x^2 - 6x + 1$ on the interval $[-2, 2]$.

Solution:

(a) We not that the function is $f(x) = \sqrt{x}$ and we linearize. $f'(x) = \frac{1}{2\sqrt{x}}$ and we choose $a = 9$ because it is a square and close to 8. Then,

$$L(x) = f(a) + f'(a)[x - a] = 3 + \frac{1}{2(3)}(x - 9) = 3 + \frac{1}{6}(x - 9)$$

Then, $L(8) = 3 + \frac{1}{6}(8 - 9) = 3 - \frac{1}{6} = \frac{17}{6}$.

(b) We start by differentiating the function.

$$f'(x) = 3x^2 + 3x - 6 = 3(x^2 + x - 2) = 3(x - 1)(x + 2)$$

and therefore, there are critical points at $x = 1, -2$. The function values at these points are $f(1) = 1 + 3/2 - 6 + 1 = 7/2 - 6 = -5/2$ and $f(-2) = (-2)^3 + 3/2(-2)^2 - 6(-2) + 1 = 11$. Lastly, we check the endpoints. We already know that $f(-2) = 11$ so the other endpoint is $f(2) = (2)^3 + 3/2(2)^2 - 6(2) + 1 = 3$. And therefore, the absolute maximum is 11 at $x = -2$ and the absolute minimum is $-\frac{5}{2}$ at $x = 1$.

5. (32 pts) Let $f(x) = \dfrac{x}{x^2 + 1}$.

(a) Find all vertical and horizontal asymptotes. If none exist, state this.

(b) Find the intervals of increase/decrease and critical points.

(c) Find intervals of concavity and points of inflection. (You can use the fact that $f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}$ to save time)

(d) Use this information to sketch the function.

Solution:

(a) There are no vertical asymptotes because the denominator is never zero. Taking the limit to find the horizontal asymptotes,

$$\lim_{x\to\infty} \frac{x}{x^2+1} = \lim_{x\to\infty} \frac{1/x}{1+1/x^2} = \frac{0}{1} = 0$$

Note that the function is odd, so the limit $x \to -\infty$ is the same. And so, there is a horizontal asymptote at $y = 0$.

(b) We find the derivative:

$$f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$$

The denominator is always positive and so we look at the numerator. Critical points occur when $-x^2 + 1 = 0$ or $x = \pm 1$. Checking the derivative at the points $-2, 0, 2$ we see that the interval of increase is $(-1, 1)$ and the interval of decrease is $(-\infty, -1) \cup (1, \infty)$.

(c) Using the given second derivative, we see that the denominator is always positive. Points of inflection occur when $2x(x^2-3) = 2x(x-\sqrt{3})(x+\sqrt{3}) = 0$ or when $x = -\sqrt{3}, 0, \sqrt{3}$. Evaluating between these points, we can see that the interval of negative concavity is $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$ and the interval of positive concavity is $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$

(d) The graph is