APPM 1350 **Exam 1** **Summer 2020**

On the front page please write your name and clearly label each problem This exam is worth 100 points and has 4 questions on both sides of this paper.

- Make sure all of your work is on separate sheets of paper. Nothing on this exam sheet will be graded. Please begin each problem on a new page.
- **Show all work and simplify your answers!** Name any theorem that you use. Answers with no justification will receive no points unless the problem explicitly states otherwise.
- Notes, papers, calculators, cell phones, and other electronic devices are not permitted, except for a computer for proctoring through Zoom.

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1. (25 pts) For this problem, let $f(x) = \dfrac{\sin(x)}{x^2 - 1}$.

   (a) What is the domain of $f(x)$?

   (b) Does $f(x)$ have any horizontal asymptotes? If not, demonstrate this. If yes, determine the equation(s) of the horizontal asymptote(s).

   (c) Does $f(x)$ have any vertical asymptotes? If not, demonstrate this. If yes, determine the equation(s) of the vertical asymptote(s).

   Solution:

   (a) Note that $\sin(x)$ and $x^2 - 1$ are continuous and so the only place where $f(x)$ is undefined is where the denominator is 0. Therefore, $f(x)$ is defined everywhere except $-1, 1$ and so the domain is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

   (b) Taking the limit $\lim\limits_{x\to\infty} \frac{\sin(x)}{x^2-1}$, we note that $-1 \le \sin(x) \le 1$ and so

   $$-\frac{1}{x^2 - 1} \le \frac{\sin(x)}{x^2 - 1} \le \frac{1}{x^2 - 1}$$

   and

   $$\lim_{x\to\infty} -\frac{1}{x^2 - 1} = \lim_{x\to\infty} \frac{1}{x^2 - 1} = 0$$

   and by the Squeeze Theorem, $\lim\limits_{x\to\infty} \frac{\sin(x)}{x^2-1} = 0$. We can similarly find that $\lim\limits_{x\to-\infty} \frac{\sin(x)}{x^2-1} = 0$ so the only horizontal asymptote is $y = 0$.

   (c) The only potential asymptotes can occur when the denominator is 0, because $\sin(x)$ is continuous. $\lim\limits_{x\to-1^-} \frac{\sin(x)}{x^2-1} = -\infty$ because the numerator is negative and non-zero and the denominator is positive and going to zero. $\lim\limits_{x\to1^+} \frac{\sin(x)}{x^2-1} = \infty$ because the numerator is positive and non-zero and the denominator is positive and going to zero. Therefore, there are vertical asymptotes at $x = -1, 1$.

2. (30 pts) Calculate the following limits, if they exist. If a limit does not exist, indicate this by writing "DNE".

(a) $\lim_{\theta \to 0} \theta \sin\left(\frac{1}{\theta}\right)$

(b) $\lim_{x \to \infty} \sqrt{x^2+2x} - \sqrt{x^2-2x}$

(c) Show that $\lim_{h \to 0} \frac{\cos(h)-1}{h} = 0$. (Hint: multiply by the conjugate and use a trig identity)

Solution:

(a) First, note that $-1 \le \sin(1/\theta) \le 1$. Then $-\theta \le \sin(1/\theta) \le \theta$ and $\lim_{\theta\to 0} -\theta = \lim_{\theta \to 0} \theta = 0$ and so, by the Squeeze Theorem,

$$\lim_{\theta \to 0} \theta \sin\left(\frac{1}{\theta}\right) = 0$$

(b) We multiply by the conjugate

$$\begin{aligned}
\lim_{x\to\infty} \sqrt{x^2+2x} - \sqrt{x^2-2x} &= \lim_{x\to\infty} (\sqrt{x^2+2x} - \sqrt{x^2-2x})\left(\frac{\sqrt{x^2+2x}+\sqrt{x^2-2x}}{\sqrt{x^2+2x}+\sqrt{x^2-2x}}\right) \\
&= \lim_{x\to\infty} \frac{(x^2+2x)-(x^2-2x)}{\sqrt{x^2+2x}+\sqrt{x^2-2x}} = \lim_{x\to\infty} \frac{4x}{\sqrt{x^2+2x}+\sqrt{x^2-2x}} \\
&= \lim_{x\to\infty} \frac{4x}{x(\sqrt{1+2/x}+\sqrt{1-2/x})} \\
&= \lim_{x\to\infty} \frac{4}{(\sqrt{1+2/x}+\sqrt{1-2/x})} = \frac{4}{\sqrt{1}+\sqrt{1}} = 2
\end{aligned}$$

(c) We start by multiplying by the conjugate:

$$\begin{aligned}
\lim_{h\to 0} \frac{\cos(h)-1}{h} &= \lim_{h\to 0} \frac{\cos(h)-1}{h}\left(\frac{\cos(h)+1}{\cos(h)+1}\right) = \lim_{h\to 0} \frac{\cos^2(h)-1}{h\,(\cos(h)+1)} \\
&\sin^2(h)+\cos^2(h) = 1 \text{ so } \cos^2(h)-1 = -\sin^2(h) \\
&= \lim_{h\to 0} \frac{-\sin^2(h)}{h\,(\cos(h)+1)} = \lim_{h\to 0} \frac{-\sin(h)}{h} \lim_{h\to 0} \frac{\sin(h)}{\cos(h)+1} \\
&= -1(0/2) = 0
\end{aligned}$$

3. (20 pts) The following questions are not related. Justify your answers and cite any theorems you use.

(a) Let $f(x) = \begin{cases} c, & x = -1 \\ \frac{x^2-x-2}{x+1}, & x \neq -1 \end{cases}$. Use the definition of continuity to determine the value of c that makes $f(x)$ continuous on $\mathbb{R}$.

(b) Given that f is a function where $f(0) = -1$ and $f(1) = 2$, is there a root in the interval $[0, 1]$? Explain why or why not (using theorems).

Solution:

(a) Rational functions are continuous except where the denominator is zero (in this case when $x = -1$). For the function to be continuous at $x = -1$, $\lim_{x\to -1} f(x) = f(-1)$. Taking the limit,

$$\lim_{x\to -1} \frac{x^2-x-2}{x+1} = \lim_{x\to -1} \frac{(x-2)(x+1)}{x+1} = \lim_{x\to -1} x - 2 = -3$$

So,

$$\lim_{x \to -1} f(x) = -3 = f(-1) = c$$

and from this, for the function to be continuous, $c = -3$.

(b) It is not guaranteed to have a root in $[0, 1]$. The Intermediate Value Theorem would guarantee a root in the interval if the function was continuous, but the function is not necessarily continuous.

TURN OVER—More problems on the back!

4. (25 pts) The following questions are not related. Justify your answers and cite any theorems you use.

(a) Find the equation of the tangent line to $f(x) = x^{1/3} + x$ at $x = 1$.

(b) Use the definition of the derivative along with the angle addition formula $\sin(A+B) = \sin(A)\cos(B) + \sin(B)\cos(A)$ and the result from Problem 2c to find $f'(\theta)$ where $f(\theta) = \sin(\theta)$.

Solution:

(a) First we use the power rule to take the derivative. $f'(x) = \frac{1}{3}x^{-2/3} + 1$ and so the slope of the tangent line is $f'(1) = 1/3(1)^{-2/3} + 1 = 4/3$. The point of tangency is at $(1, f(1))$ and $f(1) = (1)^{1/3} + 1 = 2$, and using point-slope form, the equation of the tangent line is

$$y - 2 = \frac{4}{3}(x - 1)$$

(b) Using the definition of the derivative,

$$\begin{aligned}
\frac{d}{d\theta}[\sin(\theta)] &= \lim_{h\to 0} \frac{\sin(\theta + h) - \sin(\theta)}{h} \\
&= \lim_{h\to 0} \frac{\sin(\theta)\cos(h) + \sin(h)\cos(\theta) - \sin(\theta)}{h} \\
&= \lim_{h\to 0} \cos(\theta)\frac{\sin(h)}{h} + \lim_{h\to 0} \sin(\theta)\frac{\cos(h) - 1}{h}
\end{aligned}$$

We know that $\lim\limits_{h\to 0} \frac{\sin(h)}{h} = 1$ and from Problem 2c $\lim\limits_{h\to 0} \frac{\cos(h)-1}{h} = 0$ and so,

$$\frac{d}{d\theta}[\sin(\theta)] = \cos(\theta)$$