1. (43 pts total)

(a) Evaluate the following integrals:

i. (9 pts) $\displaystyle\int \frac{(1+e^x)^2}{e^x}dx$

ii. (10 pts) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{1+\cos(x)}\,dx$

(b) Evaluate the following limits:

i. (8 pts) $\displaystyle\lim_{x\to-\infty} \tanh(2x)$

ii. (8 pts) $\displaystyle\lim_{x\to 0} x^2\cos\left(\frac{1}{\sqrt{x}}\right)$

iii. (8 pts) $\displaystyle\lim_{x\to 0^-} e^{3/x}$

Solution:

(a) Integrals:

i.

$$\begin{aligned}\int \frac{(1+e^x)^2}{e^x}dx &= \int e^{-1}(e^{2x}+2e^x+1)dx \\ &= \int e^x + 2 + e^{-x}dx \\ &= \boxed{e^x + 2x - e^{-x} + C}\end{aligned}$$

ii. With a u-substitution $u = 1+\cos(x)$ so $du = -\sin(x)dx$, we have

$$\begin{aligned}\int_0^{\pi/2} \frac{\sin(x)}{1+\cos(x)}dx &= \int_2^0 \frac{-du}{du} \\ &= \ln(u)|_0^1 \\ &= \ln(2) - \ln(1) \\ &= \boxed{\ln(2)}.\end{aligned}$$

(b) Limits:

i.

$$\begin{aligned}\lim_{x\to-\infty} \tanh(2x) &= \lim_{x\to-\infty} \frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}} \\ &= \lim_{x\to-\infty} \frac{e^{-2x}(e^{4x}-1)}{e^{-2x}(e^{4x}+1)} \\ &= \lim_{x\to-\infty} \frac{e^{4x}-1}{e^{4x}+1} \\ &= \frac{\lim_{x\to-\infty} e^{4x}-1}{\lim_{x\to-\infty} e^{4x}+1} \\ &= \frac{0-1}{0+1} = \boxed{-1}.\end{aligned}$$

ii. A squeeze theorem problem. Begin by noting that $-1 \leq \cos(1/\sqrt{x}) \leq 1$ for any $x \neq 0$. Thus,

$$-x^2 \leq x^2 \cos(1/\sqrt{x}) \leq x^2$$

and

$$0 = \lim_{x \to 0} -x^2 \leq \lim_{x \to 0} \cos(1/\sqrt{x}) \leq \lim_{x \to 0} x^2 = 0.$$

By the squeeze theorem,

$$\boxed{\lim_{x \to 0} \cos(1/\sqrt{x}) = 0.}$$

iii. Note that $\lim_{x \to 0^-} \frac{3}{x} = -\infty = \lim_{t \to -\infty} t$, so

$$\lim_{x \to 0^-} e^{3/x} = \lim_{t \to -\infty} e^t = \boxed{0}.$$

2. (22 pts total) Find the area of the largest rectangle inscribed in a right triangle with leg lengths 3 and 4 if two sides of the rectangle lie along the legs.

Solution:

(a) If we align the x-axis of a coordinate system with the length 3 leg and the y-axis with the length 4 leg and place the origin at the right-angle corner, the hypotenuse edge of the triangle is defined by the line $y = 4 - \frac{4}{3}x$. If our rectangle has one corner on the origin with base length x, the corresponding height of the rectangle is $y = 4 - \frac{4}{3}x$. Thus, the area of such a rectangle is

$$A(x) = xy = x\left(4 - \frac{4}{3}x\right) = 4x - \frac{4}{3}x^2.$$

We seek to maximize this area, so we search for critical values:

$$\frac{d}{dx}A(x) = 4 - \frac{8}{3}x = 0 \Rightarrow x = \frac{3}{2}.$$

The area corresponding to this base length is

$$A\left(\frac{3}{2}\right) = \frac{3}{2}(4-2) = \boxed{3}.$$

3. (40 pts total) The following problems are unrelated.

(a) (11 pts) Calculate y' if $y = \tan(xy)$.

(b) (13 pts) Find the equation of the tangent line to the curve of the function $f(x) = (\ln(x))^x$ at the point $(e, 1)$. Write your answer in the form $y = mx + b$. Use this linearization to estimate $\ln(3)^3$.

(c) (16 pts) Let $f(x) = \ln(2 + \ln(x))$. Determine the domain of f and f^{-1}, and find a formula for $f^{-1}(x)$.

Solution:

(a) Implicit differentiation:

$$y' = \sec^2(xy) \cdot \frac{d}{dx}(xy) \Rightarrow y' = \sec^2(xy)(xy' + y) \Rightarrow \boxed{y' = \frac{y\sec^2(xy)}{1 - x\sec^2(xy)}}.$$

(b) Rewrite $f(x) = \exp(x \ln(\ln(x)))$, so

$$\begin{aligned} f'(x) &= \exp(x\ln(\ln(x)))\frac{d}{dx}(x\ln(\ln(x))) \\ &= \exp(x\ln(\ln(x)))\left(\ln(\ln(x)) + x\frac{1}{\ln(x)}\frac{1}{x}\right) \\ &= (\ln(x))^x\left(\ln(\ln(x)) + \frac{1}{\ln(x)}\right). \end{aligned}$$

So $f'(e) = (\ln(e))^e\left(\ln(\ln(e)) + \frac{1}{\ln(e)}\right) = 1^e(0+1) = 1$. The slope of the tangent line is 1. Therefore, the equation of the tangent line is given by $y - 1 = 1(x-e) \Rightarrow \boxed{y = x - e + 1}$. To estimate $\ln(3)^3$, note that this is the value of $f(3)$. Since $f(x) \approx L(x) = x - e + 1$ near 3, we have $\boxed{\ln(3)^3 \approx 3 - e + 1 = 4 - e}$.

(c) To be in the domain of f, we require $x > 0$ and $2 + \ln(x) > 0 \Leftrightarrow x > e^{-2}$.

Thus, $\boxed{\text{the domain of } f \text{ is } [e^{-2}, \infty)}$.

Note the range of f is $\mathbb{R}$, so we can conclude that $\boxed{\text{the domain of } f^{-1} \text{ is } \mathbb{R}}$.

To find a formula for f^{-1}, note that

$$\begin{aligned} y &= \ln(2 + \ln(x)) \\ \Rightarrow e^y &= 2 + \ln(x) \\ \Rightarrow e^y - 2 &= \ln(x) \\ \Rightarrow 7x &= e^{e^y - 2}. \end{aligned}$$

So $\boxed{f^{-1}(x) = e^{e^x - 2}}$. We can also deduce from this formula that the domain of f^{-1} is $\mathbb{R}$.

4. (45 pts total) Let $g(x) = \displaystyle\int_{-x^3}^{8} e^{t^2}\, dt$.

   (a) (12 pts) Calculate $g(-2)$ and $g'(-2)$.

   (b) (15 pts) Find the intervals on which the graph of $g(x)$ is concave up/down.

   (c) (10 pts) Demonstrate that g is one-to-one.

   (d) (8 pts) Calculate $(g^{-1})'(0)$.

Solution:

(a) $\boxed{g(-2) = \displaystyle\int_8^8 e^{x^2}\, dx = 0}$.

By FTC, taking $u = -x^3$

$$g'(x) = \frac{d}{dx}\int_{-x^3}^{8} e^{t^2}\, dt = \frac{d}{du}\left(\int_u^8 e^{t^2}\, dt\right)\frac{du}{dt} = -e^{u^2}\frac{du}{dt} = 3x^2 e^{x^6}.$$

Thus, $\boxed{g'(2) = 12e^{64}}$.

(b) $g''(x) = 6xe^{x^6} + 18x^7e^{x^6} = 6xe^{x^6}(1 + 3x^6)$; $g''(x) = 0$ only if $x = 0$, so we can take any negative and positive sample points to determine the sign of g'' on $(-\infty, 0)$ and $(0, \infty$, respectively. $g''(1) = 24e > 0$ $g''(-1) = -24e < 0$.

$\boxed{\text{So } g \text{ is concave up when } x > 0 \text{ and concave down when } x < 0}$. $x = 0$ is a point of inflection.

(c) From part 1: note that $g^{-1}(0) = 2$, since we showed that $g(2) = 0$. Thus,

$$(g^{-1})'(0) = \frac{1}{g'(g^{-1}(0))} = \frac{1}{g'(2)} = \boxed{\frac{1}{12e^{64}}}$$
