1. Evaluate the following integrals:

   (a) (7 pts) $\displaystyle\int \frac{x^2}{(x^3+1)^2}\,dx$

   (b) (8 pts) $\displaystyle\int_0^2 x\sqrt{x+1}\,dx$

   (c) (7 pts) $\displaystyle\int \cos(x)\sin(x)\,dx$

   (d) (8 pts) $\displaystyle\int_0^{\pi/4} \tan(x)\sec(x)\,dx$

   Solution:

   (a) Let $u = x^3+1$, so $du = 3x^2dx$, or $x^2dx = \frac{1}{3}du$. Then we may rewrite our integral:

   $$\int \frac{x^2dx}{(x^3+1)^2} = \frac{1}{3}\int \frac{du}{u^2} = -\frac{1}{3}u^{-1} + C = -\frac{1}{3}\frac{1}{x^3+1} + C.$$

   (b) Let $u = x+1$, so $x = u-1$ and $du = dx$. We have:

   $$\begin{aligned}\int_0^2 x\sqrt{x+1}dx &= \int_1^3 (u-1)\sqrt{u}du \\ &= \int_1^3 u^{3/2} - u^{1/2}du \\ &= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\Big|_1^3 \\ &= \frac{2}{5}3^{5/2} - \frac{2}{3}3^{3/2} - (\frac{2}{5} - \frac{2}{3}) \\ &= \sqrt{3}\left(\frac{2}{5}3^2 - \frac{2}{3}3\right) - \frac{6-10}{15} \\ &= \sqrt{3}\left(\frac{18}{5} - 2\right) + \frac{4}{15} \\ &= \frac{24\sqrt{3}+4}{15}.\end{aligned}$$

   (c) Let $u = \sin(x)$, then $du = \cos(x)$. Then

   $$\int \sin(x)\cos(x)dx = \int udu = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2(x) + C.$$

   Note also that a substitution of $u = \cos(x)$ is possible, with a resulting answer of $-\frac{1}{2}\cos^2(x) + C$.

   (d)

   $$\int_0^{\pi/4} \tan(x)\sec(x)dx = \sec(x)|_0^{\pi/4} = \sec(\pi/4) - \sec(0) = \sqrt{2} - 1.$$

2. The following problems are unrelated. Fully justify your answers and cite any theorems you use.

   (a) (6 pts) Suppose we're using Newton's method to determine the root of the function $f(x) = x^3 - 2x - 5$. Use Newton's method to determine the second approximation x_2 of the root, given an initial guess of $x_1 = 2$.

(b) (8 pts) Suppose that $f(x)$ is integrable on the interval $[1, 8]$. Suppose further that $\int_1^6 2f(x)dx = 10$ and $\int_6^8 f(x)dx = -2$. Evaluate $\int_1^8 3f(x)dx$.

(c) (8 pts) Evaluate the following limit:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\left[\left(\frac{k}{n}\right)^3 + 1\right].$$

(d) (6 pts) Show that the value of $\int_0^1 \sqrt{1+\cos(x)}dx$ cannot possibly be 2.

Solution:

(a)

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \left.\frac{x^3 - 2x - 5}{3x^2 - 2}\right|_{x=2} = 2 - \frac{8-4-5}{12-2} = 2 + \frac{1}{10} = 2.1.$$

(b) $\int_1^6 2f(x)dx = 10 \Rightarrow 2\int_1^6 f(x)dx = 10 \Rightarrow \int_1^6 f(x)dx = 5$. Therefore,

$$\int_1^8 3f(x)dx = 3\int_1^8 f(x)dx = 3\left[\int_1^6 f(x)dx + \int_6^8 f(x)dx\right] = 3(5-2) = 9.$$

(c)

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\left[\left(\frac{k}{n}\right)^3 + 1\right] &= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\left[\frac{k^3}{n^3} + 1\right] \\
&= \lim_{n\to\infty} \frac{1}{n^4}\sum_{k=1}^{n} k^3 + \frac{1}{n}\sum_{k=1}^{n} 1 \\
&= \lim_{n\to\infty} \frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2 + \frac{1}{n}(n) \\
&= \lim_{n\to\infty} \frac{1}{4}\left(\frac{n+1}{n}\right)\left(\frac{n+1}{n}\right) + 1 \\
&= \lim_{n\to\infty} \frac{1}{4}\frac{1+\frac{1}{n}}{1}\frac{1+\frac{1}{n}}{n} + 1 \\
&= \frac{1}{4}(1)(1) + 1 = \frac{5}{4}.
\end{aligned}$$

(d) Note that $1 + \cos(x) \le 2$, so $\sqrt{1+\cos(x)} \le \sqrt{2}$. Thus,

$$\int_0^1 \sqrt{1+\cos(x)}dx \le \int_0^1 \sqrt{2}dx = \sqrt{2} < 2.$$

3. The velocity of a particle is given by $v(t) = t^2 - \sin(t)$ for $0 \le t \le \pi$.

(a) (7 pts) Estimate the net displacement of the particle using a midpoint approximation with two equispaced subintervals.

(b) (8 pts) Determine the average velocity of the particle.

(c) (5 pts) Write (but do not evaluate) an integral to calculate the total distance traveled by the particle.

Solution:

(a) The midpoint approximation uses rectangles with base dimension $\pi/2$ and sample points $\pi/4$ and $3\pi/4$:

$$s \approx \frac{\pi}{2}v(\pi/4)+\frac{\pi}{2}v(3\pi/4) = \frac{\pi}{2}\left(\frac{\pi^2}{16}-\frac{\sqrt{2}}{2}\right)+\frac{\pi}{2}\left(\frac{9\pi^2}{16}-\frac{\sqrt{2}}{2}\right) = \frac{\pi}{2}\left(\frac{10\pi^2}{16}-\sqrt{2}\right) = \frac{5\pi^3}{16}-\frac{\sqrt{2}\pi}{2}.$$

(b) The average velocity of the particle is

$$v_{\text{avg}} = \frac{1}{\pi-0}\int_0^\pi v(t)dt = \frac{1}{\pi}\int_0^\pi t^2-\sin(t)dt = \frac{1}{\pi}\left[\frac{1}{3}t^3+\cos(t)\right]_0^\pi = \frac{1}{\pi}\left[\frac{\pi^3}{3}-1-(1)\right] = \frac{\pi^2}{3}-\frac{2}{\pi}.$$

(c)

$$d = \int_0^\pi |v(t)|dt = \int_0^\pi |t^2-\sin(t)|dt.$$

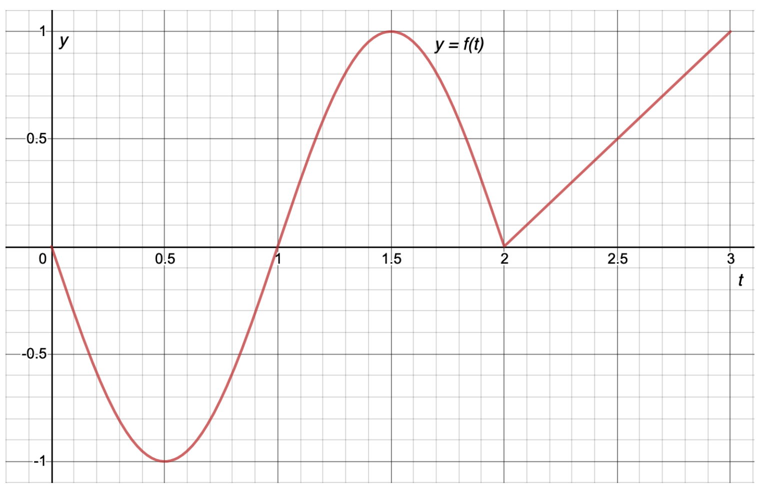

Figure 1: The graph of $y = f(t)$ for problem 4.

4. Figure 1 depicts the graph of a function $f(t)$, for $0 \leq t \leq 3$. For the following problems, define $g(x) = 2 + \displaystyle\int_{3/2}^{x} f(t)dt$.

(a) (8 pts) Determine the values of $g(3/2)$, $g'(3/2)$, and $g''(3/2)$.

(b) (7 pts) Determine the interval(s) on which $g(x)$ is increasing.

(c) (7 pts) Determine the interval(s) on which $g(x)$ is concave up.

Solution:

(a) By FTC1, $g'(x) = f(x)$ and $g''(x) = f'(x)$. So
$g(3/2) = 2 + \int_{3/2}^{3/2} f(t)dt = 2 + 0 = 2.$
$g'(3/2) = f(3/2) = 1.$
$g''(3/2) = f'(3/2) = 0$, since f has a horizontal tangent line at this point.

(b) g is increasing when $g'(x) = f(x) > 0$, which is on the intervals $(1, 2) \cup (2, 3)$.

(c) g is concave up when $g''(x) = f'(x) > 0$, or when f is increasing. This occurs on the intervals $(0.5, 1.5) \cup (2, 3)$.

The following identities may be useful:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2.$$