1. (20 pts) For each of the following, find the requested information:

   (a) $y = \sec^2(3x)$; $\frac{dy}{dx}$.

   (b) $y = \dfrac{x}{\sqrt{2-x^2}}$; $\frac{dy}{dx}$.

   (c) $y^2(2-x) = x^3$; the slope of the tangent line to the curve at the point (1,1).

   Solution:

   (a) Chain rule: $y' = 2 \cdot 3\sec(3x)\frac{d}{dx}\sec(3x) = 6\tan(3x)\sec(3x)$.

   (b) Quotient rule and chain rule: $y' = \dfrac{\sqrt{2-x^2} - x\left(\frac{-x}{\sqrt{2-x^2}}\right)}{2-x^2} = \dfrac{2-x^2+x^2}{(2-x^2)^{3/2}} = \frac{2}{(2-x^2)^{3/2}}$

   (c) Implicit differentiation: $2yy'(2-x) - y^2 = 3x^2 \Rightarrow y' = \dfrac{3x^2+y^2}{2y(2-x)}$. At (1,1), this evaluates to $y'|_{(1,1)} = \frac{3+1}{2} = 2$.

2. (24 pts) The following problems are unrelated. Fully justify your answers and cite any theorems you use.

   (a) A small child flies a kite at a height of 30 m. The wind carries the kite horizontally away from her at a rate of 6 m/s. How fast must she let out the string when the kite is 60 m away from her?

   (b) Find two positive numbers whose sum is 18 and whose product is as large as possible.

   Solution:

   (a) Let the height of the kite be y; let the horizontal distance from the kite be x; let d be the distance from the kite to the child (this is the current length of string). These quantities obey the relation $x^2 + y^2 = d^2$. We wish to find dd/dt when $dx/dt = 6$ m/s, $y = 30$, $dy/dt = 0$, and $d = 60$. Differentiating our Pythagorean relation with respect to t yields

   $$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2d\frac{dd}{dt} \Leftrightarrow \frac{dd}{dt} = \frac{1}{d}\left[x\frac{dx}{dt} + y\frac{dy}{dt}\right].$$

   We need only calculate x: $x = \sqrt{d^2 - y^2} = \sqrt{60^2 - 30^2} = 30\sqrt{2^2 - 1^2} = 30\sqrt{3}$. Thus,

   $$\frac{dd}{dt} = \frac{1}{60}\left[30\sqrt{3} \cdot 6 + 30(0)\right] = 3\sqrt{3} \text{ m/s}.$$

   (b) Let the two numbers be labeled x and y. We seek to maximize the product $p = xy$, respecting the sum $x + y = 18$. Then $y = 18 - x$, and $p = p(x) = x(18-x) = 18x - x^2$. Searching for critical values, we look for solutions of $p'(x) = 18 - 2x = 0 \Rightarrow x = 9$. Then $y = 9$ also. This is the extreme value of p since $p'(x) > 0$ for $x < 9$ and $p'(x) < 0$ for $x > 9$. The two numbers are $x, y = 9$.

3. (28 pts) Let $f(x) = \dfrac{8}{x^2-4}$, with $f'(x) = -\dfrac{16x}{(x^2-4)^2}$ and $f''(x) = \dfrac{16(3x^2+4)}{(x^2-4)^3}$.

   (a) Find all asymptotes for f. Justify your answers using the appropriate limits.

(b) Find the intervals on which f is increasing and decreasing, respectively. Justify your answers.

(c) Find the local maximum and minimum values for the function f. Justify your answer.

(d) Find the intervals of concavity and the points of inflection for the function f. Justify your answers.

(e) Sketch the graph of f, using your results from (a)-(d) to inform your sketch. Label all asymptotes, local maxima/minima, and inflection points on the graph.

Solution:

(a) This curve has a horizontal asymptote of $y = 0$, since

$$\lim_{x \to \pm\infty} \frac{8}{x^2 - 4} = 0.$$

The curve also has a vertical asymptote at $x = 2$, since

$$\lim_{x \to 2^-} \frac{8}{\underbrace{x^2 - 4}_{<0, \to 0}} = -\infty.$$

By even symmetry, the curve also has a vertical asymptote at $x = -2$.

(b) Note that the denominator of $f'(x)$ is positive for every x, so we need only determine the sign of the numerator, $-16x$. When $x < 0$ (and $x \neq -2$), $-16x > 0$, so $f'(x) > 0$. On the other hand, when $x > 0$ (and $x \neq 2$), $-16x < 0$, so $f'(x) < 0$. Therefore, f is increasing on $(-\infty, -2) \cup (-2, 0)$ and decreasing on $(0, 2) \cup (2, \infty)$.

(c) By Fermat's theorem, local extreme occur when $f'(x) = 0$ or $f'(x)$ does not exist. $f'(x)$ fails to exist only at the vertical asymptotes $x = \pm 2$, so we need only investigate the situation where $f'(x) = 0$. This occurs at $x = 0$. By work from part (b), $f(x)$ is increasing for $x < 0$ and decreasing for $x > 0$, so $f(0) = -2$ is a local max.

(d) The graph of f is concave up (CU) on intervals where $f''(x) > 0$ and concave down (CD) on intervals where $f''(x) < 0$. The numerator of $f''(x)$ is always positive, so the sign of $f''(x)$ can only switch across the vertical asymptotes $x = \pm 2$. We can sample points from the three resulting subintervals:

$(-\infty, -2)$: $f''(-3) = \frac{16(3(3)+4)}{(9-4)^3} > 0 \Rightarrow$ CU.

$(-2, 2)$: $f''(0) = \frac{16(4)}{-4^3} < 0 \Rightarrow$ CD.

$(2, \infty)$: $f''(3) = \frac{16(3(3)+4)}{(9-4)^3} > 0 \Rightarrow$ CU.

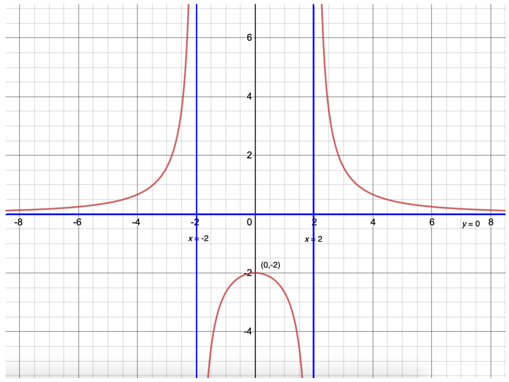
There are no points of inflection: although the concavity of the function changes across $x = \pm 2$, the change does not occur at a **point on the graph** of $f(x)$.

(e) The graph is at the top of the next page.

4. (28 pts) The following problems are unrelated. Fully justify your answers and cite any theorems you use.

(a) Let $f(x) = (1 + x)^k$ for some constant k. Determine the tangent line approximation to f at $(0, 1)$. Use this to estimate $\sqrt{1.2}$ and $\sqrt[3]{1.3}$.

(b) Consider the curve $y = \dfrac{x^2 - 3}{2x - 4}$. Find the equation of the slant asymptote to the curve. Write down (**but do not evaluate**) the limit that you would use to demonstrate that the line you found is a slant asymptote.



(c) Suppose that f is continuous and differentiable on the interval $[-1, 4]$. Suppose further that $f(-1) = 1$ and $f'(x) \leq 4$. What is the largest possible value of $f(4)$?

Solution:

(a) $f'(x) = k(1+x)^{k-1}$, so $f'(0) = k$. Our tangent line approximation satisfies $y-1 = k(x-0) \Rightarrow y = kx + 1$.

$$\sqrt{1.2} = (1 + 0.2)^{1/2} \approx 1 + \frac{1}{2}(0.2) = 1.1.$$

$$\sqrt[3]{1.3} = (1 + 0.3)^{1/3} \approx 1 + \frac{1}{3}(0.3) = 1.1.$$

(b) Carrying out polynomial long division:

$$\begin{array}{r|l} & \frac{1}{2}x + 1 \\ \hline 2x - 4 & x^2 \qquad - 3 \\ & -x^2 + 2x \\ \hline & \quad 2x - 3 \\ & \quad -2x + 4 \\ \hline & \qquad 1 \end{array}$$

we obtain the line $y = \frac{1}{2}x+1$ as our slant asymptote. The limit we would evaluate to demonstrate that this line is a slant asymptote is

$$\lim_{x \to \pm\infty} \frac{x^2 - 3}{2x - 4} - \left(\frac{1}{2}x + 1\right) = 0.$$

(c) The mean value theorem applies to this function, so we know that

$$\frac{f(4) - f(-1)}{4 - (-1)} = f'(c) \Leftrightarrow f(4) = 5f'(c) + f(-1)$$

for some c in $(-1, 4)$. If $f'(c) \leq 4$, then

$$f(4) \leq 4(5) + f(-1) = 20 + 1 = 21.$$