1. (26 pts) For this problem, let $f(x) = \dfrac{x-3}{\sqrt{x}}$.

   (a) What is the domain of $f(x)$?

   (b) Where is $f(x)$ continuous?

   (c) Does $f(x)$ have any horizontal asymptotes? If not, demonstrate this. If yes, determine the equation(s) of the horizontal asymptote(s).

   (d) Does $f(x)$ have any vertical asymptotes? If not, demonstrate this. If yes, determine the equation(s) of the vertical asymptote(s).

   (e) Determine the equation of the tangent line to the curve $y = f(x)$ at the point $(1, -2)$.

   Solution:

   (a) (4 pts) The domain of f is anywhere that $x \geq 0$ (to avoid the square root of a negative number) and also $x \neq 0$ (to avoid division by zero). Hence, the domain is $(0, \infty)$.

   (b) (4 pts) Since both $x-3$ and $\sqrt{x}$ are continuous on the domain of f, and the quotient of continuous functions is continuous (wherever the denominator is nonzero), f is continuous on $(0, \infty)$.

   (c) (5 pts) We check $\lim_{x\to\pm\infty} f(x)$. Note that f is undefined for $x < 0$, so $\lim_{x\to-\infty} f(x)$ doesn't exist. Then
   $$\lim_{x\to\infty} \frac{x-3}{\sqrt{x}} = \lim_{x\to\infty} \sqrt{x} - \frac{3}{\sqrt{x}} = \infty - 0 = \infty.$$
   Because this limit does not approach a number, there are no horizontal asymptotes.

   (d) (5 pts) f is continuous everywhere on its domain, so it has no vertical asymptotes on $(0, \infty)$. However, it could still have an asymptote at $x = 0$. We check with the following limit:
   $$\lim_{x\to 0^+} \frac{x-3}{\sqrt{x}} = \lim_{x\to 0^+} \sqrt{x} - \frac{3}{\sqrt{x}} = 0 - \infty = -\infty.$$
   Because this limit approaches infinity, we say that $x = 0$ is a vertical asymptote.

   (e) (8 pts) The slope of the tangent line is the derivative of the function at the point. We can rewrite the given function as $f(x) = \sqrt{x} - 3x^{-1/2}$, so
   $$f'(x) = \frac{1}{2}x^{-1/2} + \frac{3}{2}x^{-3/2} \Rightarrow f'(1) = \frac{1}{2}(1) + \frac{3}{2}(1) = 2.$$
   Thus, the equation of the tangent line is
   $$y - (-2) = 2(x-1) \Leftrightarrow y = 2x - 4.$$

2. (28 pts) Calculate the following limits, if they exist. If a limit does not exist, indicate this by writing "DNE".

   (a) $\displaystyle\lim_{x\to 0} \sin(x + \cos(x))$

   (b) $\displaystyle\lim_{x\to 3} \left[\frac{1}{x-3} - \frac{5}{x^2 - x - 6}\right]$

   (c) $\displaystyle\lim_{x\to 0} x^2 \sin\left(\cos\left(\frac{1}{x}\right)\right)$

   (d) $\displaystyle\lim_{x\to 0^-} \frac{x}{|x|}$

Solution:

(a) (7 pts) $\sin(x)$ and $\cos(x)$ are both continuous, everywhere, so

$$\lim_{x\to 0} \sin(x+\cos(x)) = \sin\left(\lim_{x\to 0} x + \cos\left(\lim_{x\to 0} x\right)\right) = \sin(0+\cos(0)) = \sin(1).$$

(b) (7 pts) In pursuit of a common denominator, we find

$$\begin{aligned}
\lim_{x\to 3} \frac{1}{x-3} - \frac{5}{x^2-x+6} &= \lim_{x\to 0} \frac{1}{x-3} - \frac{5}{(x-3)(x+2)} \\
&= \lim_{x\to 3} \frac{x+2}{(x-3)(x+2)} - \frac{5}{(x-3)(x+2)} \\
&= \lim_{x\to 3} \frac{x-3}{(x-3)(x+2)} \\
&= \lim_{x\to 3} \frac{1}{x+2} \\
&= \frac{1}{5}
\end{aligned}$$

(c) (7 pts) Note that $-1 \le \sin(\theta) \le 1$ for any θ, so in fact we also have $-1 \le \sin(\cos(1/x)) \le 1$. Multiplying across by x^2, we have $-x^2 \le x^2 \sin(\cos(1/x)) \le x^2$. Now observe that $\lim_{x\to 0} x^2 = \lim_{x\to 0} -x^2 = 0$, so by the squeeze theorem we can conclude that

$$\lim_{x\to 0} x^2 \sin(\cos(1/x)) = 0.$$

(d) (7 pts) Since we're investigating a left-hand limit toward zero, we may assume that all values of x are negative. Thus, we can rewrite $|x| = -x$. We then have

$$\lim_{x\to 0^-} \frac{x}{|x|} = \lim_{x\to 0^-} \frac{x}{-x} = \lim_{x\to 0^-} -1 = -1.$$

3. (26 pts) The following questions are not related. Justify your answers and cite any theorems you use.

(a) Let $r(s) = \begin{cases} as^2+9, & \text{if } s \le 2 \\ \dfrac{s^2+s-6}{s-2}, & \text{if } s > 2. \end{cases}$ Use the *definition of continuity* to determine the value of a that makes $r(s)$ continuous on $\mathbb{R}$.

(b) Suppose that a continuous function $f(x)$ satisfies $0 < f(x) < 1$ for every x in $[0,1]$. Consider $g(x) = f(x) - x$; demonstrate that $g(x)$ has at least one root in the interval $(0,1)$. Remember to cite any theorems that you may use. (*Hint*: Note that g is also continuous).

(c) Consider $g(t) = t^2 - 3t + 2$. Use the *limit definition of the derivative* to calculate $g'(2)$.

(d) Consider $h(x) = \dfrac{\tan(x)}{x^3}$. Is $h(x)$ odd, even, or neither? Justify your answer.

Solution:

(a) (7 pts) The definition of continuity says that a function is continuous at a point c if $\lim_{s\to c} r(x) = r(c)$. Inspecting this function, we see that it's a polynomial for $s \le 2$ (which is continuous everywhere) and a rational function for $s > 2$ which is continuous everywhere except $s = 2$.

Thus, r is continuous everwhere except possibly at $s = 2$. To determine an a to make the function continuous here, we require

$$\lim_{s\to 2^-} r(s) = \lim_{s\to 2^+} r(s) = r(2).$$

$r(2) = 4a + 9$, and certainly $\lim_{x\to 2^-} r(s) = 4a + 9$ (by the continuity of polynomials). So we just need

$$4a + 9 = \lim_{s\to 2^-} r(s) = \lim_{s\to 2^-} \frac{s^2 + s - 6}{s - 2} = \lim_{s\to 2^-} \frac{(s-2)(s+3)}{s-2} = \lim_{s\to 2^-} s + 3 = 5.$$

We solve for a using the resulting equation $4a + 9 = 5$, which yields $a = -1$.

(b) (6 pts) Note that $g(0) = f(0) > 0$ and $g(1) = f(1) - 1 < 0$. Since g is continuous and $g(0) > 0$ and $g(1) < 0$, the **Intermediate Value Theorem** guarantees a c between 0 and 1 such that $g(c) = 0$.

(c) (7 pts)

$$\begin{aligned}
\lim_{h\to 0} \frac{g(2+h) - g(2)}{h} &= \lim_{h\to 0} \frac{(2+h)^2 - 3(2+h) + 2 - [2^2 - 3(2) + 2]}{h} \\
&= \lim_{h\to 0} \frac{4 + 4h + h^2 - 6 - 3h + 2 - 4 + 6 - 2}{h} \\
&= \lim_{h\to 0} \frac{h + h^2}{h} \\
&= \lim_{h\to 0} 1 + h \\
&= 1.
\end{aligned}$$

(d) (6 pts) Let's check what $h(-x)$ looks like. Note that

$$h(-x) = \frac{\tan(-x)}{(-x)^3} = \frac{\frac{\sin(-x)}{\cos(-x)}}{-x^3} = \frac{\frac{-\sin(x)}{\cos(x)}}{-x^3} = \frac{\tan(x)}{x^3} = h(x).$$

So h is even.

4. (20 pts) For each of the following statements, determine if the statement is **TRUE** or **FALSE**. If the statement is **TRUE**, briefly explain why. If the statement is **FALSE**, briefly explain why or provide an example that shows the statement is false.

(a) If a differentiable function $f(x)$ is periodic with period T, then $f'(x)$ is also periodic with period T. (Reminder: f is periodic with period T if $f(x + T) = f(x)$ for each x in its domain.)

(b) If $\lim_{x\to\infty} f(x) = L$, then L is not in the range of f.

(c) The curve $y = \frac{1}{x-1}$ has a vertical asymptote given by the line $y = 1$.

(d) If $f'(a)$ exists, then $\lim_{x\to a} f(x) = f(a)$.

Solution:

(a) (5 pts) **TRUE.** If a function is periodic, the slope of the tangent line will have the same slope from period to period. One can also show this from definition by writing

$$f'(a+T) = \lim_{h\to 0} \frac{f(a+T+h) - f(a+T)}{h} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = f'(a).$$

(b) (5 pts) **FALSE.** The simplest example I can think of is the constant function $f(x) = 0$. For this function the range of f contains *only* $\lim_{x\to\infty} f(x)$.

(c) (5 pts) **FALSE**. The vertical asymptote is given by the line $x = 1$.

(d) (5 pts) **TRUE**. $f'(a)$ exists means that f is differentiable at a. $\lim_{x\to a} f(x) = f(a)$ means that f is continuous at a. Differentiability implies continuity.
