1. [30 pts] Let $f(x) = \dfrac{e^x}{4 - e^x}$.

   (a) Find all asymptotes, if any, of the graph of $f(x)$. Full justification requires the appropriate use of limits.

   (b) Find the linearization of $f(x)$ at $x = 0$.

   (c) Find the area under the graph of $f(x)$ between $x = 0$ and $x = 1$.

SOLUTION:

(a) $y = 0$ and $y = 1$ are horizontal asymptotes since

$$\lim_{x\to\infty} \frac{e^x}{4-e^x} \overset{\text{LH}}{=} \lim_{x\to\infty} \frac{e^x}{-e^x} = \lim_{x\to\infty} -1 = -1 \quad \text{and} \quad \lim_{x\to-\infty} \frac{e^x}{4-e^x} = \frac{0}{4-0} = 0$$

Note that $4 - e^x = 0 \implies e^x = 4 \implies x = \ln 4$. $x = \ln 4$ is a vertical asymptote since

$$\lim_{x\to\ln 4^+} \frac{e^x}{4-e^x} \to \frac{e^{\ln 4}}{4 - e^{\ln 4}} \to \frac{4}{0^-} \implies \lim_{x\to\ln 4^+} \frac{e^x}{4-e^x} = -\infty$$

(b) $f'(x) = \dfrac{(4-e^x)(e^x) - e^x(-e^x)}{(4-e^x)^2} = \dfrac{4e^x - e^{2x} + e^{2x}}{(4-e^x)^2} = \dfrac{4e^x}{(4-e^x)^2} \implies f'(0) = \dfrac{4e^0}{(4-e^0)^2} = \dfrac{4}{9}$. Furthermore, $f(0) = \dfrac{e^0}{(4-e^0)} = \dfrac{1}{3}$ so that the linearization is

$$L(x) = f(0) + f'(0)(x-0) = \frac{1}{3} + \frac{4}{9}x$$

(c) The area under the graph is given by

$$\int_0^1 \frac{e^x}{4-e^x}\,\mathrm{d}x \overset{u=4-e^x}{=} \int_3^{4-e} \frac{-1}{u}\,\mathrm{d}u = \ln|u|\Big|_{4-e}^{3} = \ln|3| - \ln|4-e| = \ln\frac{3}{4-e}$$

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2. [35 pts] Consider the function $g(x) = \displaystyle\int_{-1}^{2x} \sqrt{1+\sin t}\,\mathrm{d}t$

   (a) [5 pts] Find $g(-1/2)$.

   (b) [10 pts] Will $g(x)$ attain a maximum value on the interval $[0, \pi/2]$? Explain briefly.

   (c) [10 pts] Let c be the smallest positive critical number of $g(x)$. Find c and determine, with justification, whether or not $g(c)$ is a relative extremum.

   (d) [10 pts] Determine, with justification, if $g(x)$ has an inflection point in the interval $(0, \pi/2)$.

SOLUTION:

(a) $g(-1/2) = \displaystyle\int_{-1}^{2\left(-\frac{1}{2}\right)} \sqrt{1+\sin t}\,\mathrm{d}t = \int_{-1}^{-1} \sqrt{1+\sin t}\,\mathrm{d}t = 0$

(b) Yes. $g(x)$ is differentiable for all values of x and therefore continuous for all values of x. This means that $g(x)$ is continuous on the closed interval $[0, \pi/2]$. Therefore, as a consequence of the Extreme Value Theorem, $g(x)$ will attain a maximum on the interval.

(c) Using the Fundamental Theorem with the chain rule we have

$$g'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_{-1}^{2x} \sqrt{1 + \sin t}\, \mathrm{d}t = 2\sqrt{1 + \sin 2x}$$

$g'(x)$ exists everywhere so critical points will only occur where $g'(x) = 0$. Thus, considering only positive values of x,

$$1 + \sin 2x = 0 \implies \sin 2x = -1 \implies 2x = 3\pi/2, 7\pi/2, 9\pi/2, \dots \implies x = 3\pi/4, 7\pi/4, 9\pi/4, \dots$$

The smallest of these gives the critical point of $c = 3\pi/4$. Since $\sqrt{1 + \sin 2x} \geq 0$ for all x, $g'(x)$ never changes sign so that $g(3\pi/4)$ is not a relative extremum.

(d) We need to analyze $g''(x) = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(2\sqrt{1 + \sin 2x}\right) = \dfrac{2\cos 2x}{\sqrt{1 + \sin 2x}}$ in the interval $(0, \pi/2)$ and see if it changes sign there. $g''(x)$ will be zero if the numerator vanishes but the denominator does not. For this to occur

$$2\cos 2x = 0 \implies \cos 2x = 0 \implies 2x = \pi/2 \implies x = \pi/4$$

Evaluating the denominator at $x = \pi/4$ yields $\sqrt{2}$. Indeed, the denominator is always positive on the given interval so the sign of $g''(x)$ will be determined solely by the sign of its numerator. Since $\cos 2(0) = 1 > 0$ the graph of $g(x)$ is concave up on $(0, \pi/4)$ and because $\cos 2(\pi/2) = -1 < 0$, the graph of $g(x)$ is concave down on $(\pi/4, \pi/2)$. Since the concavity changes at $x = \pi/4$, and since $g(x)$ is continuous at $\pi/4$, $(\pi/4, g(\pi/4))$ is an inflection point of the graph of $g(x)$.

3. [65 pts] The following problems are unrelated.

(a) [10 pts] Find $(f^{-1})'(2)$ where $f(x) = x^3 + 2x - 1$.

(b) [6 pts] Let $a > 0$ be a constant. If $x = a \sin\theta$, find $\tan\theta$ in terms of x and a.

(c) [28 pts] Find the following limits.

i. $\displaystyle\lim_{x\to 0} (1 - 3x)^{1/x}$ ii. $\displaystyle\lim_{x\to\infty} \cos^{-1}\left(\frac{1 + x^2}{1 + 2x^2}\right)$ iii. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n^2}$ iv. $\displaystyle\lim_{x\to 1} \frac{\sqrt{x+1} - \sqrt{2}}{x - 1}$

(d) [21 pts] Evaluate the following integrals.

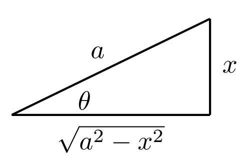
i. $\displaystyle\int_{1/3}^{\sqrt{3}/3} \frac{1}{1 + 9x^2}\, \mathrm{d}x$ ii. $\displaystyle\int \frac{x}{\sqrt{5x + 1}}\, \mathrm{d}x$ iii. $\displaystyle\int \frac{\sinh w}{1 + \sinh^2 w}\, \mathrm{d}w$

SOLUTION:

(a) Begin by finding $f^{-1}(2)$ by solving $2 = x^3 + 2x - 1$. This can be done by inspection giving $x = 1 \implies f^{-1}(2) = 1$. Next, $f'(x) = 3x^2 + 2$ yielding $f'(f^{-1}(2)) = f'(1) = 5$. Finally then

$$(f^{-1})'(2) = \frac{1}{f'\left(f^{-1}(2)\right)} = \frac{1}{5}$$

(b) $x = a \sin\theta \implies \sin\theta = \dfrac{x}{a}$ which allows us to build the following triangle.



From this it follows that $\tan\theta = \dfrac{x}{\sqrt{a^2 - x^2}} = \dfrac{x\sqrt{a^2 - x^2}}{a^2 - x^2}$.

(c) i. Direct substitution yields the indeterminate form 1^∞.

$$\lim_{x\to 0}(1-3x)^{1/x} = e^{\left[\lim\limits_{x\to 0}\frac{\ln(1-3x)}{x}\right]} \overset{\text{LH}}{=} e^{\left[\lim\limits_{x\to 0}\frac{\frac{-3}{1-3x}}{1}\right]} = e^{-3}$$

ii. Since $\cos^{-1} x$ is continuous, we can write

$$\lim_{x\to\infty}\cos^{-1}\left(\frac{1+x^2}{1+2x^2}\right) = \cos^{-1}\left[\lim_{x\to\infty}\left(\frac{1+x^2}{1+2x^2}\right)\right] \overset{\text{LH}}{=} \cos^{-1}\left[\lim_{x\to\infty}\left(\frac{2x}{4x}\right)\right] = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

iii. Note that this is a Riemann sum, so we can use the definition of the definite integral to evaluate the limit

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{n^2} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{n}\left(\frac{1}{n}\right) = \int_0^1 x\,\mathrm{d}x = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}$$

If the Riemann sum isn't obvious, use the brute force method:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{n^2} = \lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{n} i = \lim_{n\to\infty}\left(\frac{1}{n^2}\right)\frac{n(n+1)}{2} = \frac{1}{2}\lim_{n\to\infty}\left(\frac{n+1}{n}\right) = \frac{1}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = \frac{1}{2}$$

iv. Note that this is the definition of $f'(1)$ with $f(x)=\sqrt{x+1}$.

$$\lim_{x\to 1}\frac{\sqrt{x+1}-\sqrt{2}}{x-1} = \left.\left(\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x+1}\right)\right|_{x=1} = \left.\frac{1}{2\sqrt{x+1}}\right|_{x=1} = \frac{1}{2\sqrt{2}}$$

If the definition of the derivative is not obvious, then use the brute force method:

$$\begin{aligned}\lim_{x\to 1}\frac{\sqrt{x+1}-\sqrt{2}}{x-1} &= \lim_{x\to 1}\frac{\sqrt{x+1}-\sqrt{2}}{x-1}\left(\frac{\sqrt{x+1}+\sqrt{2}}{\sqrt{x+1}+\sqrt{2}}\right) = \lim_{x\to 1}\frac{x+1-2}{(x-1)\left(\sqrt{x+1}+\sqrt{2}\right)} \\ &= \lim_{x\to 1}\frac{1}{\sqrt{x+1}+\sqrt{2}} = \frac{1}{2\sqrt{2}}\end{aligned}$$

Or, if it is noted that direct substitution yields the indeterminate form $0/0$, you can use l'Hospital's Rule:

$$\lim_{x\to 1}\frac{\sqrt{x+1}-\sqrt{2}}{x-1} \overset{\text{LH}}{=} \lim_{x\to 1}\frac{\frac{1}{2\sqrt{x+1}}}{1} = \frac{1}{2\sqrt{2}}$$

(d) i.

$$\begin{aligned}\int_{1/3}^{\sqrt{3}/3}\frac{1}{1+9x^2}\,\mathrm{d}x &= \int_{1/3}^{\sqrt{3}/3}\frac{1}{1+(3x)^2}\,\mathrm{d}x \overset{u=3x}{=} \frac{1}{3}\int_1^{\sqrt{3}}\frac{1}{1+u^2}\,\mathrm{d}u = \left.\frac{1}{3}\tan^{-1}u\right|_1^{\sqrt{3}} \\ &= \frac{1}{3}\left(\tan^{-1}\sqrt{3}-\tan^{-1}1\right) = \frac{1}{3}\left(\frac{\pi}{3}-\frac{\pi}{4}\right) = \frac{\pi}{36}\end{aligned}$$

ii.

$$\begin{aligned}\int\frac{x}{\sqrt{5x+1}}\,\mathrm{d}x &\overset{u=5x+1}{=} \frac{1}{5}\int\frac{u-1}{5\sqrt{u}}\,\mathrm{d}u = \frac{1}{25}\int\left(u^{1/2}-u^{-1/2}\right)\mathrm{d}u = \frac{1}{25}\left(\frac{2}{3}u^{3/2}-2u^{1/2}\right)+C \\ &= \frac{2}{25}\left[\frac{1}{3}(5x+1)^{3/2}-(5x+1)^{1/2}\right]+C \\ &= \frac{2}{25}\sqrt{5x+1}\left(\frac{5}{3}x+\frac{1}{3}-1\right)+C = \frac{2}{75}(5x-2)\sqrt{5x+1}+C\end{aligned}$$

iii. Recall that $\cosh^2 w - \sinh^2 w = 1 \implies 1 + \sinh^2 w = \cosh^2 w$.

$$\int \frac{\sinh w}{1+\sinh^2 w}\,\mathrm{d}w = \int \frac{\sinh w}{\cosh^2 w}\,\mathrm{d}w \overset{u=\cosh w}{=} \int \frac{1}{u^2}\,\mathrm{d}u = -\frac{1}{u} + C = -\frac{1}{\cosh w} + C = -\operatorname{sech} w + C$$

Alternatively,

$$\int \frac{\sinh w}{\cosh^2 w}\,\mathrm{d}w = \int \left(\frac{\sinh w}{\cosh w}\right)\left(\frac{1}{\cosh w}\right)\mathrm{d}w = \int \tanh w \operatorname{sech} w\,\mathrm{d}w = -\operatorname{sech} w + C$$

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4. [10 pts] A bug crawls along the curve $y = \cosh x$. If the y–coordinate of the bug is changing at a rate of -4 mm/sec, how fast is its x–coordinate changing when $x = \ln 3$? Write your final answer without any hyperbolic functions, using appropriate definitions.

SOLUTION:
The bug's coordinates are both functions of t. Since the bug is crawling on the curve, the rate of change of its x– and y–coordinates are related through

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \sinh x \frac{\mathrm{d}x}{\mathrm{d}t} \implies \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{\sinh x}\frac{\mathrm{d}y}{\mathrm{d}t}$$

We are given that $\frac{\mathrm{d}y}{\mathrm{d}t} = -4$ and $x = \ln 3$. Substituting and simplifying yields

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{\sinh(\ln 3)}(-4) = \frac{-4}{\frac{e^{\ln 3} - e^{-\ln 3}}{2}} = \frac{-8}{3 - \frac{1}{3}} = \frac{-8}{\frac{8}{3}} = -3 \text{ mm/sec}$$

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5. [10 pts] Suppose an experimental population of fruit flies in your lab grows at a rate that is proportional to the number of flies present. Your lab assistant tells you that there were 100 flies after 2 days and 300 flies after 4 days. Approximately how many flies were in the original population?

SOLUTION:
Since the population grows at a rate proportional to the population at any time, if $P(t)$ is the population at time t (days) we have $P(t) = P(0)e^{kt}$. Using the given data, $100 = P(0)e^{2k}$ and $300 = P(0)e^{4k}$. Solving the first of these for $P(0)$ and substituting into the second gives

$$P(0) = 100e^{-2k} \implies 300 = 100e^{-2k}e^{4k} \implies 3 = e^{2k} \implies \ln 3 = 2k \implies k = \frac{\ln 3}{2}$$

Thus

$$P(0) = 100e^{-2(\ln 3)/2} = 100e^{-\ln 3} = 100e^{\ln \frac{1}{3}} = 100\left(\frac{1}{3}\right) \approx 33$$

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POTENTIALLY HELPFUL FORMULAS

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$