1. [35 pts] Consider the function $f(x) = x^4 - 6x^2$.

   (a) [2 pts] What is the domain of $f(x)$?

   (b) [3 pts] Is $f(x)$ odd, even or neither? Justify your answer algebraically.

   (c) [3 pts] Find all the x–intercepts and y–intercepts of $f(x)$, if any.

   (d) [2 pts] Find all the asymptotes of $f(x)$, if any.

   (e) [2 pts] Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

   (f) [2 pts] Find $f'(x)$. Check your answer very carefully before proceeding.

   (g) [2 pts] Find $f''(x)$. Check your answer very carefully before proceeding.

   (h) [3 pts] Where is $f(x)$ increasing and where is $f(x)$ decreasing? Write your answer using interval notation.

   (i) [3 pts] Find all local extrema of $f(x)$, if it possesses any.

   (j) [3 pts] Where is $f(x)$ concave up and where is $f(x)$ concave down? Write your answer using interval notation.

   (k) [3 pts] Find all inflection points of $f(x)$, if it possesses any.

   (l) [7 pts] Sketch the graph of $f(x)$. Label all intercepts, relative extrema (if any) and inflection points (if any).

<u>**SOLUTION:**</u>

(a) $f(x)$ is a polynomial so its domain is all real numbers $\mathbb{R}$ or $(-\infty, \infty)$

(b) $f(-x) = (-x)^4 - 6(-x)^2 = x^4 - 6x^2 = f(x) \implies f(x)$ is even (symmetric with respect to y–axis)

(c) $f(0) = 0$ so y–intercept is $(0, 0)$. $x^4 - 6x^2 = 0 \implies x^2(x^2 - 6) = x^2(x + \sqrt{6})(x - \sqrt{6}) = 0 \implies x = 0, \pm\sqrt{6}$ so the x–intercepts are $(0, 0)$, $(-\sqrt{6}, 0)$ and $(\sqrt{6}, 0)$.

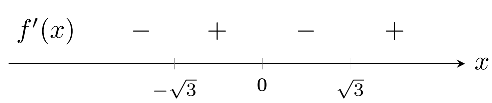
(d) There are no asymptotes since $f(x)$ is a polynomial.

(e) $\displaystyle\lim_{x\to\infty} x^4\left(1 - \frac{6}{x^2}\right) = \lim_{x\to-\infty} x^4\left(1 - \frac{6}{x^2}\right) = \infty$

(f) $f'(x) = 4x^3 - 12x$

(g) $f''(x) = 12x^2 - 12$

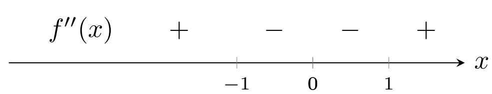
(h) $f'(x) = 4x(x^2 - 3) = 4x(x + \sqrt{3})(x - \sqrt{3})$ so the critical points are $x = 0, \pm\sqrt{3}$ and the sign of f' is shown below:



From the chart, $f(x)$ is increasing on $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$ and decreasing on $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$

(i) $f(-\sqrt{3}) = (-\sqrt{3})^4 - 6(-\sqrt{3})^2 = 9 - 18 = -9$ and $f(\sqrt{3}) = (\sqrt{3})^4 - 6(\sqrt{3})^2 = 9 - 18 = -9$ are local minima and $f(0) = 0$ is a local maximum.

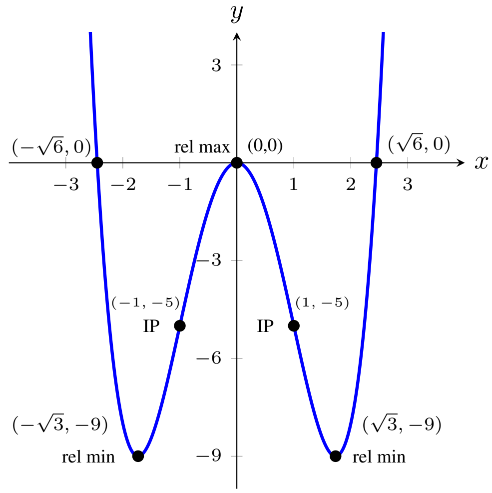
(j) $f''(x) = 12(x^2 - 1) = 12(x + 1)(x - 1)$ and the sign of f'' is shown below:



From the chart, $f(x)$ is concave up on $(-\infty, -1) \cup (1, \infty)$ and concave down on $(-1, 1)$.

(k) Inflection points are $(-1, f(-1)) = (-1, -5)$ and $(1, f(1)) = (1, -5)$.

(l) Sketch



2. [18 pts] Consider the function $p(x) = x^3 - 3x + 4$.

   (a) Write the equation that uses Newton's method to find the root(s) of $p(x)$.

   (b) Using your answer to part (a), if $x_1 = 0$, find x_2.

   (c) Suppose x_1 is chosen such that $x_2 = -1$. What is x_3 in this case? Explain briefly.

SOLUTION:

(a) $p'(x) = 3x^2 - 3$ so Newton's method is $x_{n+1} = x_n - \dfrac{p(x_n)}{p'(x_n)} = x_n - \dfrac{x_n^3 - 3x_n + 4}{3x_n^2 - 3}$.

(b) $x_2 = 0 - \dfrac{0^3 - 3(0) + 4}{3(0^2) - 3} = \dfrac{4}{3}$

(c) x_3 cannot be computed as the denominator in the equation for Newton's method would be 0.

3. [15 pts] The following problems are unrelated.

   (a) If $f(x) = 10 - x^2$, $1 \le x \le 5$, evaluate the Riemann sum with $n = 4$, taking the sample points to be right endpoints.

   (b) Evaluate $\displaystyle\int_2^{10} |x - 7|\, \mathrm{d}x$ by interpreting it in terms of areas.

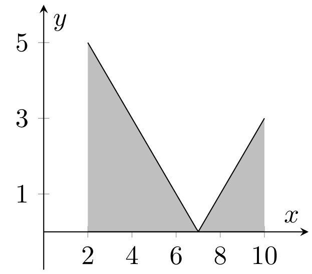
   (c) Evaluate $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n} \left(\frac{i}{n}\right)^2$.

SOLUTION:

(a) We have $\Delta x = \dfrac{b-a}{n} = \dfrac{5-1}{4} = 1$ so that $x_i^\star = 1 + i\Delta x = 1 + i,\ \ n = 1, 2, 3, 4$. The Riemann sum becomes

$$\begin{aligned}\sum_{i=1}^{n} f(x_i^\star)\Delta x &= \sum_{i=1}^{4} f(1+i)(1) = f(2) + f(3) + f(4) + f(5) \\ &= (10 - 2^2) + (10 - 3^2) + (10 - 4^2) + (10 - 5^2) = -14\end{aligned}$$

(b) Begin by graphing the integrand and note that the integral represents the shaded area comprised of two triangles.



The area of the left triangle is $\frac{1}{2}(7-2)(5) = \frac{25}{2}$ and the area of the right triangle is $\frac{1}{2}(10-7)(3) = \frac{9}{2}$. Thus

$$\int_2^{10} |x-7|\,\mathrm{d}x = \frac{25}{2} + \frac{9}{2} = \frac{34}{2} = 17$$

(c)

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left(\frac{i}{n}\right)^2 = \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \frac{i^2}{n^2} = \lim_{n\to\infty} \frac{3}{n}\left(\frac{1}{n^2}\right) \sum_{i=1}^{n} i^2 = \lim_{n\to\infty} \frac{3}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \lim_{n\to\infty} \frac{1}{2n^2}\left(2n^2+3n+1\right) = \lim_{n\to\infty}\left(1 + \frac{3}{2n} + \frac{1}{2n^2}\right) = 1+0+0 = 1$$

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4. [12 pts] If 1200 cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box. Be sure to show that this is the maximum volume.

SOLUTION:

Let the length of one side of the base of the box be x and let the height of the box be y. Then the box's volume is $V = x^2 y$. The surface area of the box is the area of the base (x^2) plus 4 times the area of one side (xy), or $S = x^2 + 4xy$. We are told that this must equal 1200 so that $x^2 + 4xy = 1200 \implies y = \dfrac{1200 - x^2}{4x}$. We use this to eliminate y in the equation for volume obtaining $V(x) = x^2\left(\dfrac{1200 - x^2}{4x}\right) = \dfrac{1}{4}\left(1200x - x^3\right)$. For this problem, the domain of $V(x)$ is $x > 0$ and we wish to maximize $V(x)$ on this domain. To that end,

$$V'(x) = \frac{1}{4}\left(1200 - 3x^2\right) = 0 \implies 1200 = 3x^2 \implies x = 20 \ \text{(positive root only since } x > 0)$$

is the only critical point of $V(x)$. Noting that $V''(x) = -\frac{3}{2}x$ is always negative for $x > 0$ shows that $V(x)$ is concave down on that interval implying that $V(20)$ is the absolute maximum value there. So $V(20) = \frac{1}{4}\left[1200(20) - 20^3\right] = 4000$ cm^3 is the largest possible volume. (Note that $V'(x) > 0$ for $0 < x < 20$ and $V'(x) < 0$ for $x > 20$, also showing that $V(x)$ attains an absolute maximum at $x = 20$.)

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5. [20 pts] The following problems are unrelated.

(a) [6 pts] Let $g(x) = \dfrac{x}{x^2+1}$. Noting that $g'(x) = \dfrac{1-x^2}{(x^2+1)^2}$ and $g''(x) = \dfrac{2x\left(x^2-3\right)}{\left(x^2+1\right)^3}$, use the **second** derivative test to classify the relative extrema of $g(x)$.

(b) [6 pts] Find the most general antiderivative of $f(w) = \dfrac{5}{\sqrt[3]{w^2}} + \dfrac{2}{\sqrt{w^3}}$.

(c) [8 pts] A particle is moving with an acceleration $a(t) = 10\sin t + 3\cos t$. When time $t = 0$, the particle's position is 0 and when $t = 2\pi$ its position is 12. Find its position when $t = 5\pi/2$.

SOLUTION:

(a) First find the critical points of the function. These are points in the domain of $g(x)$ where $g'(x) = 0$ or $g'(x)$ does not exist. Note that the domain of $g(x)$ is all real numbers. Since the denominator is always positive, $g'(x)$ exists everywhere and critical points will only occur where $g'(x)$ vanishes. This happens if the numerator $1 - x^2 = (1-x)(1+x) = 0 \implies x = \pm 1$. The critical points of $g(x)$ are 1 and -1. To apply the second derivative test, we evaluate $g''(x)$ at each critical point and determine the sign.

$$g''(1) = \frac{2(1)(1^2-3)}{(1^2+1)^3} = -\frac{1}{2} < 0 \implies g(1) = \frac{1}{2} \text{ is a relative maximum}$$

$$g''(-1) = \frac{2(-1)((-1)^2-3)}{((-1)^2+1)^3} = \frac{1}{2} > 0 \implies g(-1) = -\frac{1}{2} \text{ is a relative minimum}$$

(b) $f(w) = \dfrac{5}{\sqrt[3]{w^2}} + \dfrac{2}{\sqrt{w^3}} = 5w^{-2/3} + 2w^{-3/2}$ so that

$$F(w) = 5\frac{w^{-2/3+1}}{(-2/3+1)} + 2\frac{w^{-3/2+1}}{(-3/2+1)} + C = 15w^{1/3} - 4w^{-1/2} + C = 15\sqrt[3]{w} - \frac{4}{\sqrt{w}} + C$$

(c)

$$v(t) = \int (10\sin t + 3\cos t)\,\mathrm{d}t = -10\cos t + 3\sin t + C_1$$

$$\implies s(t) = \int (-10\cos t + 3\sin t + C_1)\,\mathrm{d}t = -10\sin t - 3\cos t + C_1 t + C_2$$

$$s(0) = -10\sin 0 - 3\cos 0 + C_1(0) + C_2 = 0 \implies -3 + C_2 = 0 \implies C_2 = 3$$

$$s(2\pi) = -10\sin 2\pi - 3\cos 2\pi + C_1(2\pi) + 3 = 12 \implies C_1 = \frac{6}{\pi}$$

Thus $s(t) = -10\sin t - 3\cos t + \frac{6}{\pi}t + 3$ and $s\left(\frac{5\pi}{2}\right) = -10\sin\frac{5\pi}{2} - 3\cos\frac{5\pi}{2} + \frac{6}{\pi}\frac{5\pi}{2} + 3 = -10 + 15 + 3 = 8$

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Potentially helpful formulas

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$