1. [30 pts] For the given function, find the indicated derivative. Simplify your final answers, writing them without negative exponents.

   a. $p(t) = \dfrac{1}{\sqrt{3-t}} + 2\pi^7, \quad \dfrac{\mathrm{d}^2 p}{\mathrm{d}t^2}$ b. $y(\theta) = \sin(\cos 6\theta), \quad y'\left(\dfrac{\pi}{4}\right)$ c. $f(x) = \dfrac{1 + x\tan x}{\sqrt{x}}, \quad f'(x)$

   SOLUTION:

   (a)

$$p(t) = (3-t)^{-1/2} + 2\pi^7 \implies \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{1}{2}(3-t)^{-3/2}(-1) + 0 = \frac{1}{2}(3-t)^{-3/2}$$
$$\implies \frac{\mathrm{d}^2 p}{\mathrm{d}t^2} = \frac{1}{2}\left(-\frac{3}{2}\right)(3-t)^{-5/2}(-1) = \frac{3}{4}(3-t)^{-5/2} = \frac{3}{4(3-t)^{5/2}} = \frac{3}{4\sqrt{(3-t)^5}}$$

   (b)

$$y'(\theta) = \cos(\cos 6\theta)(\cos 6\theta)' = \cos(\cos 6\theta)(-\sin 6\theta)(6\theta)' = -6(\sin 6\theta)\cos(\cos 6\theta))$$
$$\implies y'\left(\frac{\pi}{4}\right) = -6\sin\left(\frac{6\pi}{4}\right)\cos\left[\cos\left(\frac{6\pi}{4}\right)\right] = -6\sin\left(\frac{3\pi}{2}\right)\cos\left[\cos\left(\frac{3\pi}{2}\right)\right] = -6(-1)\cos 0 = 6$$

   (c)

$$f(x) = \frac{1 + x\tan x}{x^{1/2}} \implies f'(x) = \frac{x^{1/2}\left(x\sec^2 x + \tan x\right) - (1 + x\tan x)\left(\frac{1}{2}x^{-1/2}\right)}{\left(x^{1/2}\right)^2}$$
$$= \frac{x^{3/2}\sec^2 x + x^{1/2}\tan x - \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2}\tan x}{x} = \frac{x^{3/2}\sec^2 x + \frac{1}{2}x^{1/2}\tan x - \frac{1}{2}x^{-1/2}}{x}$$
$$= \frac{x^{-1/2}\left(x^2\sec^2 x + \frac{1}{2}x\tan x - \frac{1}{2}\right)}{x} = \frac{2x^2\sec^2 x + x\tan x - 1}{2x^{3/2}}$$

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2. [10 pts] Use differentials/linearization to approximate the value of $\sqrt[3]{26}$.

   SOLUTION:
   The function we need to linearize is $f(x) = \sqrt[3]{x} = x^{1/3}$ and we want to linearize it at a point close to 26, so we let $a = 27$. Then $f(a) = f(27) = 27^{1/3} = 3$. With $f'(x) = \dfrac{1}{3x^{2/3}}$ we have $f'(a) = f'(27) = \dfrac{1}{3(27^{2/3})} = \dfrac{1}{3(9)} = \dfrac{1}{27}$. The linearization of f is $L(x) = f(27) + f'(27)(x - 27)$ or

$$L(x) = 3 + \frac{1}{27}(x - 27) \implies \sqrt[3]{26} \approx L(26) = 3 + \frac{1}{27}(26 - 27) = 3 - \frac{1}{27} = \frac{80}{27}$$

   Using differentials, we have

$$f(26) - f(27) = \sqrt[3]{26} - \sqrt[3]{27} \approx \mathrm{d}y = f'(27)\,\mathrm{d}x = \frac{1}{27}(-1) = -\frac{1}{27} \implies \sqrt[3]{26} = \sqrt[3]{27} - \frac{1}{27} = 3 - \frac{1}{27} = \frac{80}{27}$$

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3. [10 pts] The curve $Ax^2 + By^2 - 3y = 2$ has the tangent line $y = 1 - \frac{2}{5}(x-1)$ at the point $(1,1)$. Assuming that the given equation determines y implicitly as a differentiable function of x, find the constants A and B.

SOLUTION:

Using implicit differentiation,

$$2Ax + 2Byy' - 3y' = 0 \implies (2By-3)y' = -2Ax \implies y' = \frac{-2Ax}{2By-3}$$

We know at the point $(1,1)$ on the graph that the tangent line's slope is $-\dfrac{2}{5}$ giving

$$-\frac{2}{5} = \frac{-2A(1)}{2B(1)-3} \implies \frac{2}{5} = \frac{2A}{2B-3} \implies 4B - 6 = 10A$$

Since $(1,1)$ lies on the graph,

$$A(1^2) + B(1^2) - 3(1) = 2 \implies A + B = 5 \implies B = 5 - A$$

Combining the above yields

$$4(5-A) - 6 = 10A \implies 20 - 4A - 6 = 10A \implies 14 = 14A \implies A = 1 \implies B = 5 - 1 = 4$$

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4. The following questions are not related.

(a) [20 pts] Consider the function $f(x) = x^3 + \dfrac{48}{x}$.

i. Find all the critical points of $f(x)$.

ii. Find, if they exist, the absolute/global extrema of $f(x)$ on $[1,3]$.

(b) [10 pts] Answer true or false, justifying your answer: There must be a point in $(0,3)$ where the instantaneous rate of change of the function $f(x) = |x-1|$ equals the average rate of change of the function over the interval $[0,3]$.

(c) [10 pts] Does $g(x) = x^4 - x^2 + 3$ satisfy the hypotheses of Rolle's Theorem on the interval $[0,1]$? If so, find the c guaranteed by Rolle's theorem. If not, explain why not.

SOLUTION:

(a) $f(x) = x^3 + 48x^{-1}$

i. $f'(x) = 3x^2 - \dfrac{48}{x^2}$. This does not exist at $x = 0$. However, $x = 0$ is not in the domain of f and thus $x = 0$ is not a critical point of the function.

$$f'(x) = 3x^2 - \frac{48}{x^2} = 0 \implies 3x^4 = 48 \implies x^4 = 16 \implies x = \pm 2$$

Both $x = -2$ and $x = 2$ are in the domain of $f(x)$ and are thus critical points of the function.

ii. Since $f(x)$ is continuous on $[1,3]$ (sum of two continuous functions) and $[1,3]$ is closed, the Extreme Value Theorem states that f will attain a maximum and minimum value on the interval. To find them, evaluate f at the critical point found above that lies in the given interval ($x = 2$), namely $f(2) = 2^3 + \frac{48}{2} = 32$. Evaluating f at the endpoints of the interval yields $f(1) = 1^3 + \frac{48}{1} = 49$ and $f(3) = 3^3 + \frac{48}{3} = 43$. Therefore the absolute maximum is 49 and the absolute minimum is 32.

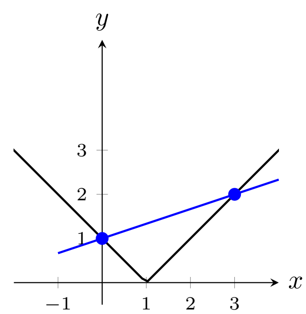
(b) False. $f(x)$ is not differentiable at $x = 1$ (cusp). Note that

$$f(x) = \begin{cases} x - 1 & x \geq 1 \\ -x + 1 & x < 1 \end{cases} \implies f'(x) = \begin{cases} 1 & x > 1 \\ \text{DNE} & x = 1 \\ -1 & x < 1 \end{cases}$$

Thus the hypotheses of the Mean Value Theorem are not satisfied. This means that there is no guarantee of the existence of a c in $(0, 3)$ such that

$$\text{Instantaneous rate of change } = f'(c) = \frac{|3-1| - |0-1|}{3-0} = \frac{1}{3} = \text{ average rate of change}$$

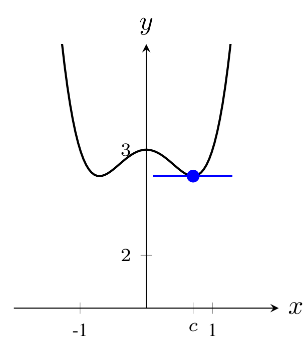
Indeed, the slope of the tangent line, if it exists, is either 1 or -1 and never $\frac{1}{3}$. Here is a sketch to illustrate. The blue line is the secant line the slope of which is the average rate of change of f over the interval $[0, 3]$. Note that there is no tangent line parallel to the secant line.



(c) $g(x)$ is a polynomial and therefore continuous and differentiable on $(-\infty, \infty)$. Thus $g(x)$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$. Furthermore, $g(0) = g(1) = 3$. So the hypotheses of Rolle's Theorem are satisfied, guaranteeing the existence of a point c in $(0, 1)$ where $g'(c) = 0$. To find this point, set the derivative to zero and solve.

$$g'(x) = 4x^3 - 2x = 2x(2x^2 - 1) = 0 \implies x = 0, x = \pm\frac{\sqrt{2}}{2} \implies c = \frac{\sqrt{2}}{2}$$

Note that the other two roots of $g'(x)$ are not in the interval $(0, 1)$. Here is a sketch of what is going on.



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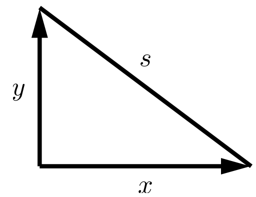
5. [10 pts] Two hikers begin walking from the same point at constant speeds. One travels north at 1.5 mph and the other travels east. Two hours later, the distance between them is 4 miles and is increasing at 3 mph. How fast is the eastbound hiker walking?

SOLUTION:



Using the notation in the figure, we are given $\frac{\mathrm{d}s}{\mathrm{d}t} = 3$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 1.5$, and we are asked to find $\frac{\mathrm{d}x}{\mathrm{d}t}$. From the Pythagorean Theorem $x^2 + y^2 = s^2$ which, upon differentiation, yields

$$2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t} = 2s\frac{\mathrm{d}s}{\mathrm{d}t} \implies \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{x}\left(s\frac{\mathrm{d}s}{\mathrm{d}t} - y\frac{\mathrm{d}y}{\mathrm{d}t}\right)$$

After 2 hours, the northbound hiker has gone $(1.5 \text{ mph})(2 \text{ hours}) = 3$ miles $= y$ and we are told that $s = 4$ miles. Consequently, at this time $x = \sqrt{s^2 - y^2} = \sqrt{4^2 - 3^2} = \sqrt{7}$ and

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{\sqrt{7}}\left[4\,(3) - 3\left(\frac{3}{2}\right)\right] = \frac{15}{2\sqrt{7}} = \frac{15\sqrt{7}}{14} \text{ mph}$$

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