1. Let \( f(x) = 2 + \frac{2 - 2x}{x^2 - 5x + 4} \)

   (a) [6 pts] What is the domain of \( f \)? Write your answer using interval notation.

   (b) [9 pts] Does \( f \) have any removable discontinuities? Justify your answer using limits.

   (c) [9 pts] Find the asymptotes of \( f \). Justify your answer using limits.

   **SOLUTION:**

   (a) \( f(x) = 2 + \frac{2 - 2x}{(x-1)(x-4)} \). The denominator vanishes if \( x = 1 \) or \( x = 4 \). Thus the domain of \( f(x) \) is

   \((-\infty, 1) \cup (1, 4) \cup (4, \infty))

   (b) Removable discontinuities can occur at points \( a \) where \( f(x) \) is not defined but the limit of \( f(x) \) as \( x \) approaches \( a \) exists. \( f \) is not defined at \( x = 1 \) and \( x = 4 \) so we consider limits at these points.

   \[
   \lim_{x \to 1} f(x) = \lim_{x \to 1} \left( 2 + \frac{2 - 2x}{x^2 - 5x + 4} \right) = \lim_{x \to 1} \left( 2 + \frac{2(1-x)}{(x-1)(x-4)} \right) = \lim_{x \to 1} \left( 2 + \frac{-2}{x-4} \right) = 2 + \frac{-2}{1-4} = \frac{8}{3} \implies x = 1 \text{ is a removable discontinuity}
   \]

   \[
   \lim_{x \to 4} f(x) = \lim_{x \to 4} \left( 2 + \frac{2 - 2x}{x^2 - 5x + 4} \right) = \lim_{x \to 4} \left( 2 + \frac{2(1-x)}{(x-1)(x-4)} \right) = \lim_{x \to 4} \left( 2 + \frac{-2}{x-4} \right) \rightarrow \text{does not exist} \implies x = 4 \text{ is not a removable discontinuity}
   \]

   (c) Candidates for vertical asymptotes are points where \( f(x) \) is not defined, namely \( x = 1 \) and \( x = 4 \). We have already shown that \( x = 1 \) is a removable discontinuity. Checking \( x = 4 \),

   \[
   \lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \left( 2 + \frac{2 - 2x}{x^2 - 5x + 4} \right) = \lim_{x \to 4^+} \left( 2 + \frac{2(1-x)}{(x-1)(x-4)} \right) = \lim_{x \to 4^+} \left( 2 + \frac{-2}{x-4} \right) \rightarrow 2 + \frac{-2}{0^+} = -\infty \implies x = 4 \text{ is a vertical asymptote}
   \]

   We could also have used the left-hand limit at \( x = 4 \) to show this.

   To check for horizontal asymptotes we need to find the limits at plus and minus infinity.

   \[
   \lim_{x \to \infty} \left( 2 + \frac{2 - 2x}{x^2 - 5x + 4} \right) = \lim_{x \to \infty} 2 + \lim_{x \to \infty} \frac{2 - 2x}{x^2 - 5x + 4} = 2 + \lim_{x \to \infty} \frac{2 - 2x}{x^2 - 5x + 4} \frac{1}{x^2} = 2 + 0 = 2 \implies y = 2 \text{ is a horizontal asymptote}
   \]

   Similarly,

   \[
   \lim_{x \to -\infty} \left( 2 + \frac{2 - 2x}{x^2 - 5x + 4} \right) = \lim_{x \to -\infty} 2 + \lim_{x \to -\infty} \frac{2 - 2x}{x^2 - 5x + 4} = 2 + \lim_{x \to -\infty} \frac{2 - 2x}{x^2 - 5x + 4} \frac{1}{x^2} = 2 + 0 = 2 \implies y = 2 \text{ is a horizontal asymptote}
   \]

   \[\square\]
2. [16 pts] Using the graph of \( f(x) \) in the figure below, compute the following:

\[
\begin{align*}
\text{a. } & \lim_{x \to -2^-} f(x) \\
\text{b. } & \lim_{x \to -2^+} f(x) \\
\text{c. } & \lim_{x \to -2} f(x) \\
\text{d. } & \lim_{x \to 2} f(x) \\
\text{e. } & f(2) \\
\text{f. } & \lim_{x \to 4^-} f(x) \\
\text{g. } & \lim_{x \to 4^+} f(x) \\
\text{h. } & \lim_{x \to 4} f(x)
\end{align*}
\]

\[
\text{Solution:} \\
a. -2 \\
b. 1 \\
c. Does not exist \\
d. -1 \\
e. Not defined \\
f. -\infty \text{ (or does not exist)} \\
g. -3 \\
h. Does not exist
\]

3. (a) [6 pts] What three conditions must be met for a function \( f(x) \) to be continuous at the point \( a \)?

(b) [18 pts] Determine where the following functions are continuous, writing your answer using interval notation.

i. \( f(x) = \cos(\sin(\sqrt{x})) - (x^4 - x^2 + 3) \)

ii. \( f(x) = \frac{|x - 5|}{x - 5} \)

iii. \( f(x) = \begin{cases} \cos 3x & x \neq 0 \\ 1 & x = 0 \end{cases} \)

\[y\]
\[
\begin{array}{c}
-5 \\
-4 \\
-3 \\
-2 \\
-1 \\
1 \\
2 \\
3 \\
4 \\
-4 \\
-2 \\
2 \\
3 \\
4 \\
-2 \\
-1 \\
1 \\
2 \\
3 \\
4 \\
-4 \\
-5
\end{array}
\]

\[x\]

\text{Solution:} \\
(a) 1. \( f(a) \) must be defined.

2. \( \lim_{x \to a} f(x) \) must exist.

3. \( \lim_{x \to a} f(x) = f(a) \)

(b) i. \([0, \infty)\); \( \sqrt{x} \) continuous on \([0, \infty)\), \( \sin x \) and \( \cos x \) continuous on \((-\infty, \infty)\) so \( \cos(\sin(\sqrt{x})) \) is continuous on \([0, \infty)\). \( x^4 - x^2 + 3 \) is a polynomial, continuous on \((-\infty, \infty)\) and thus continuous on \([0, \infty)\). The difference of continuous functions is continuous.

ii. \((-\infty, 5) \cup (5, \infty)\); Jump discontinuity at \( x = 5 \). Note:

\[
f(x) = \begin{cases} \\
\frac{x - 5}{x - 5} & x - 5 > 0 \\
\frac{-\left(x - 5\right)}{x - 5} & x - 5 < 0
\end{cases} = \begin{cases} \\
1 & x > 5 \\
-1 & x < 5
\end{cases}
\]

iii. \((\infty, 0) \cup (0, \infty)\); For \( x \neq 0 \) the function is the ratio of two continuous functions and is therefore continuous.

At \( x = 0 \), \( f \) is defined, \( f(0) = 1 \), but \( \lim_{x \to 0} \frac{\cos 3x}{x} \) does not exist.
4. [10 pts] The following problems are not related.

(a) Is there a value of \( x \) such that \( x^2 - \sqrt{x-1} \) equals 4? Justify your answer.

(b) Evaluate \( \lim_{x \to 0} \left( \frac{\sin 3x}{x} + \frac{6x - 9}{x^3 - 12x + 3} \right) \).

**Solution:**

(a) Since \( x - 1 \) is continuous on \((-\infty, \infty)\) it is continuous on \([1, \infty)\). \( \sqrt{x} \) is continuous on \([0, \infty)\) implying that \( \sqrt{x-1} \) is continuous on \([1, \infty)\) (composition of continuous function is continuous). Furthermore, \( x^2 \) being a polynomial is continuous on \( \mathbb{R} \) and thus continuous on \([1, \infty)\). Thus, \( f(x) = x^2 - \sqrt{x-1} \), being the difference of two continuous functions, is continuous on \([1, \infty)\). Now \( f(1) = 1^2 - \sqrt{1-1} = 1 \) and \( f(5) = 5^2 - \sqrt{5-1} = 23 \). Since \( f(x) \) is continuous on \([1, 5]\) and \( 1 < f(1) < 4 < f(5) = 23 \), the Intermediate Value Theorem guarantees the existence of a number \( c \) in \((1, 5)\) such that \( f(c) = 4 \). So, yes, there is a value of \( x \) such that \( x^2 - \sqrt{x-1} = 4 \).

(b) Note that

\[
\lim_{x \to 0} \frac{6x - 9}{x^3 - 12x + 3} = \frac{0 - 9}{0^3 - 12(0) + 3} = \frac{-9}{3} = -3
\]

Furthermore,

\[
\lim_{x \to 0} \frac{\sin 3x}{x} = \lim_{x \to 0} \frac{3 \sin 3x}{3x} = 3 \lim_{x \to 0} \frac{\sin 3x}{3x} = 3(1) = 3
\]

Thus

\[
\lim_{x \to 0} \left( \frac{\sin 3x}{x} + \frac{6x - 9}{x^3 - 12x + 3} \right) = \lim_{x \to 0} \frac{\sin 3x}{x} + \lim_{x \to 0} \frac{6x - 9}{x^3 - 12x + 3} = 3 - 3 = 0
\]

5. Let \( f(x) = \sqrt{x + 2} \). If you need to compute any derivatives, you must use the definition.

(a) [5 pts] Find the average rate of change of \( f \) over the interval \([7, 14]\). Simplify your answer. What geometric property of the graph of \( f(x) \) does this average rate of change represent?

(b) [6 pts] Find the instantaneous rate of change of \( f \) at \( x = 2 \). What geometric property of the graph of \( f(x) \) does this instantaneous rate of change represent?

(c) [6 pts] Find the slope/intercept form of the tangent line to the graph of \( y = f(x) \) at the point \( x = 2 \).

(d) [4 pts] The graph of \( f(x) \) is shown in the figure below. In your bluebook, sketch a graph of \( f'(x) \).

**Solution:**

(a) Average rate of change \( = \frac{f(14) - f(7)}{14 - 7} = \frac{\sqrt{14} + 2 - \sqrt{7} + 2}{7} = \frac{\sqrt{16} - \sqrt{7}}{7} = \frac{4 - 3}{7} = \frac{1}{7} \)

This is the slope of the secant line to the graph of \( f(x) \) between the points \((7, 3)\) and \((14, 4)\).
(b) Method 1

\[ f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{\sqrt{x + 2} - \sqrt{2} + 2}{x - 2} = \lim_{x \to 2} \frac{\sqrt{x + 2} - 2}{x - 2} \left( \frac{\sqrt{x + 2} + 2}{\sqrt{x + 2} + 2} \right) \]

\[ = \lim_{x \to 2} \frac{x + 2 - 4}{(x - 2)(\sqrt{x + 2} + 2)} = \lim_{x \to 2} \frac{x - 2}{(x - 2)(\sqrt{x + 2} + 2)} = \lim_{x \to 2} \frac{1}{\sqrt{x + 2} + 2} = \frac{1}{\sqrt{2} + 2} = \frac{1}{4} \]

Method 2

\[ f'(2) = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h} = \lim_{h \to 0} \frac{\sqrt{2 + h} + 2 - \sqrt{2} + 2}{h} = \lim_{h \to 0} \frac{\sqrt{4 + h} - 2}{h} = \lim_{h \to 0} \frac{4 + h - 2}{h} \left( \frac{\sqrt{4 + h} + 2}{\sqrt{4 + h} + 2} \right) \]

\[ = \lim_{h \to 0} \frac{4 + h - 4}{h(\sqrt{4 + h} + 2)} = \lim_{h \to 0} \frac{h}{h(\sqrt{4 + h} + 2)} = \lim_{h \to 0} \frac{1}{\sqrt{4 + h} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{4} \]

This is the slope of the tangent line to the graph of \( f(x) \) at \( x = 2 \).

(c) The slope of the tangent line is \( 1/4 \) and the point of tangency is \( (2, f(2)) = (2, 2) \). Thus the tangent line has equation \( y - 2 = \frac{1}{4}(x - 2) \implies y = \frac{1}{4}x + \frac{3}{2} \).

(d) Graph of \( f'(x) \).

6. [5 pts] In your bluebook, write **TRUE** if the statement is true and write **FALSE** if the statement is false. No justification required and no partial credit given.

(a) \( \cos 2x = 2 \) has no solutions.

(b) \( f(x) = \sqrt{x^2 + x - 6} \) and \( g(x) = \frac{1}{\sqrt{x^2 + x - 6}} \) have the same domain.

(c) If \( f(-x) = -f(x) \) for all \( x \) in the domain of the function \( f \), then the graph of \( f(x) \) is symmetric with respect to the \( x \)-axis.

(d) If a function has a jump discontinuity at a point \( c \) in its domain, then the function is not differentiable at the point \( c \).

(e) If \( \lim_{x \to 5} f(x) = 0 \) and \( \lim_{x \to 5} g(x) = 0 \), then \( \lim_{x \to 5} \frac{f(x)}{g(x)} \) does not exist.

**SOLUTION:**

(a) **TRUE** The range of \( \cos x \) is \([-1, 1]\).

(b) **FALSE** First note that \( x^2 + x - 6 = (x + 3)(x - 2) \). Then \( x^2 + x - 6 \leq 0 \) if \(-3 \leq x \leq 2 \). Thus the domain of \( f(x) \) is \((-\infty, -3] \cup [2, \infty)\) and the domain of \( g(x) \) is \((-\infty, -3) \cup (2, \infty)\).

(c) **FALSE** Odd functions are symmetric with respect to the origin.

(d) **TRUE** If a function is not continuous at a point in its domain, then it is not differentiable at that point.

(e) **FALSE** \( \frac{0}{0} \) is indeterminate.