1. Let $f(x) = 2 + \dfrac{2-2x}{x^2-5x+4}$

   (a) [6 pts] What is the domain of f? Write your answer using interval notation.

   (b) [9 pts] Does f have any removable discontinuities? Justify your answer using limits.

   (c) [9 pts] Find the asymptotes of f. Justify your answer using limits.

SOLUTION:

(a) $f(x) = 2 + \dfrac{2-2x}{(x-1)(x-4)}$. The denominator vanishes if $x = 1$ or $x = 4$. Thus the domain of $f(x)$ is

$$(-\infty, 1) \cup (1, 4) \cup (4, \infty)$$

(b) Removable discontinuities can occur at points a where $f(x)$ is not defined but the limit of $f(x)$ as x approaches a exists. f is not defined at $x = 1$ and $x = 4$ so we consider limits at these points.

$$\begin{aligned}\lim_{x\to 1} f(x) &= \lim_{x\to 1}\left(2 + \frac{2-2x}{x^2-5x+4}\right) = \lim_{x\to 1}\left(2 + \frac{2(1-x)}{(x-1)(x-4)}\right)\\ &= \lim_{x\to 1}\left(2 + \frac{-2}{x-4}\right) = 2 + \frac{-2}{1-4} = \frac{8}{3} \implies x = 1 \text{ is a removable discontinuity}\end{aligned}$$

$$\begin{aligned}\lim_{x\to 4} f(x) &= \lim_{x\to 4}\left(2 + \frac{2-2x}{x^2-5x+4}\right) = \lim_{x\to 4}\left(2 + \frac{2(1-x)}{(x-1)(x-4)}\right)\\ &= \lim_{x\to 4}\left(2 + \frac{-2}{x-4}\right) \to \text{does not exist} \implies x = 4 \text{ is not a removable discontinuity}\end{aligned}$$

(c) Candidates for vertical asymptotes are points where $f(x)$ is not defined, namely $x = 1$ and $x = 4$. We have already shown that $x = 1$ is a removable discontinuity. Checking $x = 4$,

$$\begin{aligned}\lim_{x\to 4^+} f(x) &= \lim_{x\to 4+}\left(2 + \frac{2-2x}{x^2-5x+4}\right) = \lim_{x\to 4^+}\left(2 + \frac{2(1-x)}{(x-1)(x-4)}\right)\\ &= \lim_{x\to 4^+}\left(2 + \frac{-2}{x-4}\right) \to 2 + \frac{-2}{0^+} = -\infty \implies x = 4 \text{ is a vertical asymptote}\end{aligned}$$

We could also have used the left-hand limit at $x = 4$ to show this.
To check for horizontal asymptotes we need to find the limits at plus and minus infinity.

$$\begin{aligned}\lim_{x\to\infty}\left(2 + \frac{2-2x}{x^2-5x+4}\right) &= \lim_{x\to\infty} 2 + \lim_{x\to\infty}\left(\frac{2-2x}{x^2-5x+4}\right) = 2 + \lim_{x\to\infty}\left(\frac{2-2x}{x^2-5x+4}\right)\frac{1/x^2}{1/x^2}\\ &= 2 + \lim_{x\to\infty}\left(\frac{2/x^2 - 2/x}{1 - 5/x + 4/x^2}\right) = 2 + \frac{0}{1} = 2 \implies y = 2 \text{ is a horizontal asymptote}\end{aligned}$$

Similarly,

$$\begin{aligned}\lim_{x\to-\infty}\left(2 + \frac{2-2x}{x^2-5x+4}\right) &= \lim_{x\to-\infty} 2 + \lim_{x\to-\infty}\left(\frac{2-2x}{x^2-5x+4}\right) = 2 + \lim_{x\to-\infty}\left(\frac{2-2x}{x^2-5x+4}\right)\frac{1/x^2}{1/x^2}\\ &= 2 + \lim_{x\to-\infty}\left(\frac{2/x^2 - 2/x}{1 - 5/x + 4/x^2}\right) = 2 + \frac{0}{1} = 2 \implies y = 2 \text{ is a horizontal asymptote}\end{aligned}$$

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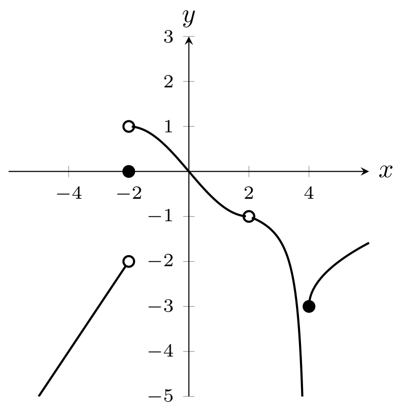
2. [16 pts] Using the graph of $f(x)$ in the figure below, compute the following:

a. $\lim_{x \to -2^-} f(x)$ b. $\lim_{x \to -2^+} f(x)$ c. $\lim_{x \to -2} f(x)$ d. $\lim_{x \to 2} f(x)$

e. $f(2)$ f. $\lim_{x \to 4^-} f(x)$ g. $\lim_{x \to 4^+} f(x)$ h. $\lim_{x \to 4} f(x)$



SOLUTION:

a. -2 b. 1 c. Does not exist d. -1 e. Not defined f. $-\infty$ (or does not exist) g. -3 h. Does not exist ■

3. (a) [6 pts] What three conditions must be met for a function $f(x)$ to be continuous at the point a?

(b) [18 pts] Determine where the following functions are continuous, writing your answer using interval notation.

i. $f(x) = \cos\left(\sin\left(\sqrt{x}\right)\right) - \left(x^4 - x^2 + 3\right)$

ii. $f(x) = \dfrac{|x-5|}{x-5}$

iii. $f(x) = \begin{cases} \dfrac{\cos 3x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

SOLUTION:

(a)

1. $f(a)$ must be defined.
2. $\lim_{x \to a} f(x)$ must exist.
3. $\lim_{x \to a} f(x) = f(a)$

(b)

i. $[0, \infty)$; $\sqrt{x}$ continuous on $[0, \infty)$, $\sin x$ and $\cos x$ continuous on $(-\infty, \infty)$ so $\cos(\sin \sqrt{x})$ is continuous on $[0, \infty)$. $x^4 - x^2 + 3$ is a polynomial, continuous on $(-\infty, \infty)$ and thus continuous on $[0, \infty)$. The difference of continuous functions is continuous.

ii. $(-\infty, 5) \cup (5, \infty)$; Jump discontinuity at $x = 5$. Note:

$$f(x) = \begin{cases} \dfrac{x-5}{x-5} & x - 5 > 0 \\ \dfrac{-(x-5)}{x-5} & x - 5 < 0 \end{cases} = \begin{cases} 1 & x > 5 \\ -1 & x < 5 \end{cases}$$

.

iii. $(\infty, 0) \cup (0, \infty)$; For $x \neq 0$ the function is the ratio of two continuous functions and is therefore continuous. At $x = 0$, f is defined, $f(0) = 1$, but $\lim_{x \to 0} \dfrac{\cos 3x}{x}$ does not exist.

■

4. [10 pts] The following problems are not related.

(a) Is there a value of x such that $x^2 - \sqrt{x-1}$ equals 4? Justify your answer.

(b) Evaluate $\lim_{x \to 0} \left(\frac{\sin 3x}{x} + \frac{6x-9}{x^3 - 12x + 3} \right)$.

SOLUTION:

(a) Since $x-1$ is continuous on $(-\infty, \infty)$ it is continuous on $[1, \infty)$. $\sqrt{x}$ is continuous on $[0, \infty)$ implying that $\sqrt{x-1}$ is continuous on $[1, \infty)$ (composition of continuous function is continuous). Furthermore, x^2 being a polynomial is continuous on $\mathbb{R}$ and thus continuous on $[1, \infty)$. Thus, $f(x) = x^2 - \sqrt{x-1}$, being the difference of two continuous functions, is continuous on $[1, \infty)$. Now $f(1) = 1^2 - \sqrt{1-1} = 1$ and $f(5) = 5^2 - \sqrt{5-1} = 23$. Since $f(x)$ is continuous on $[1, 5]$ and $1 = f(1) < 4 < f(5) = 23$, the Intermediate Value Theorem guarantees the existence of a number c in $(1, 5)$ such that $f(c) = 4$. So, yes, there is a value of x such that $x^2 - \sqrt{x-1} = 4$.

(b) Note that

$$\lim_{x \to 0} \frac{6x-9}{x^3 - 12x + 3} = \frac{0-9}{0^3 - 12(0) + 3} = \frac{-9}{3} = -3$$

Furthermore,

$$\lim_{x \to 0} \frac{\sin 3x}{x} = \lim_{x \to 0} \frac{3 \sin 3x}{3x} = 3 \lim_{x \to 0} \frac{\sin 3x}{3x} = 3(1) = 3$$

Thus $\lim_{x \to 0} \left(\frac{\sin 3x}{x} + \frac{6x-9}{x^3 - 12x + 3} \right) = \lim_{x \to 0} \frac{\sin 3x}{x} + \lim_{x \to 0} \frac{6x-9}{x^3 - 12x + 3} = 3 - 3 = 0$

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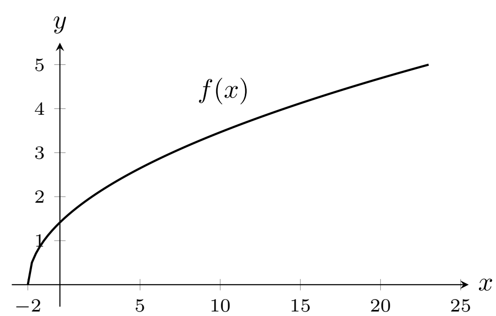
5. Let $f(x) = \sqrt{x+2}$. If you need to compute any derivatives, you must use the definition.

(a) [5 pts] Find the average rate of change of f over the interval $[7, 14]$. Simplify your answer. What geometric property of the graph of $f(x)$ does this average rate of change represent?

(b) [6 pts] Find the instantaneous rate of change of f at $x = 2$. What geometric property of the graph of $f(x)$ does this instantaneous rate of change represent?

(c) [6 pts] Find the slope/intercept form of the tangent line to the graph of $y = f(x)$ at the point $x = 2$.

(d) [4 pts] The graph of $f(x)$ is shown in the figure below. In your bluebook, sketch a graph of $f'(x)$.



SOLUTION:

(a) Average rate of change $= \frac{f(14) - f(7)}{14 - 7} = \frac{\sqrt{14+2} - \sqrt{7+2}}{7} = \frac{\sqrt{16} - \sqrt{9}}{7} = \frac{4-3}{7} = \frac{1}{7}$

This is the slope of the secant line to the graph of $f(x)$ between the points $(7, 3)$ and $(14, 4)$.

(b) Method 1

$$f'(2) = \lim_{x\to2}\frac{f(x)-f(2)}{x-2} = \lim_{x\to2}\frac{\sqrt{x+2}-\sqrt{2+2}}{x-2} = \lim_{x\to2}\frac{\sqrt{x+2}-2}{x-2} = \lim_{x\to2}\frac{\sqrt{x+2}-2}{x-2}\left(\frac{\sqrt{x+2}+2}{\sqrt{x+2}+2}\right)$$
$$= \lim_{x\to2}\frac{x+2-4}{(x-2)(\sqrt{x+2}+2)} = \lim_{x\to2}\frac{x-2}{(x-2)(\sqrt{x+2}+2)} = \lim_{x\to2}\frac{1}{\sqrt{x+2}+2} = \frac{1}{\sqrt{2+2}+2} = \frac{1}{4}$$

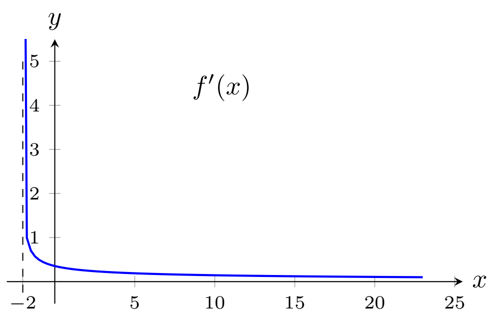
Method 2

$$f'(2) = \lim_{h\to0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to0}\frac{\sqrt{2+h+2}-\sqrt{2+2}}{h} = \lim_{h\to0}\frac{\sqrt{4+h}-2}{h} = \lim_{h\to0}\frac{\sqrt{4+h}-2}{h}\left(\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}\right)$$
$$= \lim_{h\to0}\frac{4+h-4}{h(\sqrt{4+h}+2)} = \lim_{h\to0}\frac{h}{h(\sqrt{4+h}+2)} = \lim_{h\to0}\frac{1}{\sqrt{4+h}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$$

This is the slope of the tangent line to the graph of $f(x)$ at $x = 2$.

(c) The slope of the tangent line is $1/4$ and the point of tangency is $(2, f(2)) = (2, 2)$. Thus the tangent line has equation $y - 2 = \frac{1}{4}(x-2) \implies y = \frac{1}{4}x + \frac{3}{2}$.

(d) Graph of $f'(x)$.



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6. [5 pts] In your bluebook, write **TRUE** if the statement is true and write **FALSE** if the statement is false. No justification required and no partial credit given.

(a) $\cos 2x = 2$ has no solutions.

(b) $f(x) = \sqrt{x^2+x-6}$ and $g(x) = \dfrac{1}{\sqrt{x^2+x-6}}$ have the same domain.

(c) If $f(-x) = -f(x)$ for all x in the domain of the function f, then the graph of $f(x)$ is symmetric with respect to the x–axis.

(d) If a function has a jump discontinuity at a point c in its domain, then the function is not differentiable at the point c.

(e) If $\lim_{x\to5} f(x) = 0$ and $\lim_{x\to5} g(x) = 0$, then $\lim_{x\to5}\dfrac{f(x)}{g(x)}$ does not exist.

SOLUTION:

(a) **TRUE** The range of $\cos x$ is $[-1, 1]$.

(b) **FALSE** First note that $x^2 + x - 6 = (x+3)(x-2)$. Then $x^2 + x - 6 \le 0$ if $-3 \le x \le 2$. Thus the domain of $f(x)$ is $(-\infty, -3] \cup [2, \infty)$ and the domain of $g(x)$ is $(-\infty, -3) \cup (2, \infty)$.

(c) **FALSE** Odd functions are symmetric with respect to the origin.

(d) **TRUE** If a function is not continuous at a point in its domain, then it is not differentiable at that point.

(e) **FALSE** $\dfrac{0}{0}$ is indeterminate.

■