1. (28 points) The following problems are not related.

   (a) Find the general antiderivative of $g(x) = \dfrac{e^{\sqrt{x}}}{\sqrt{x}}$.

   (b) Use logarithmic differentiation to find the derivative of $y = (x^4 + 1)^x$. *You do not need to simplify your answer.*

   (c) Find the derivative of $f(x) = \displaystyle\int_0^{\cos(x)} \sqrt{1 + t^3}\, dt$.

Solution:

(a) Setting $u = \sqrt{x}$ implies that $2du = \dfrac{dx}{\sqrt{x}}$, so

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx = 2\int e^u\, du = 2e^u + C = 2e^{\sqrt{x}} + C.$$

(b) Taking logarithms yields

$$\ln(y) = x\ln(x^4 + 1),$$

and differentiating with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \ln(x^4 + 1) + \frac{4x^4}{x^4 + 1}.$$

Solving for dy/dx in terms of x gives

$$\frac{dy}{dx} = (x^4 + 1)^x \left(\ln(x^4 + 1) + \frac{4x^4}{x^4 + 1}\right).$$

(c) $f'(x) = -\sin(x)\sqrt{1 + \cos^3(x)}$.

2. (26 points) The following problems are not related:

   (a) Find the derivative of $f(x) = \ln\left(\tan^{-1}(x)\right)$.

   (b) Evaluate the definite integral $\displaystyle\int_0^{\ln(3)} \sinh(x)\cosh(x)\, dx$, and fully simplify your answer.

   (c) Determine the value of the limit $\displaystyle\lim_{x\to 0^+} x^2 \ln(x^2)$.

Solution:

(a) $f'(x) = \left(\dfrac{1}{1 + x^2}\right)\left(\dfrac{1}{\tan^{-1}(x)}\right)$

(b) Making $u = \sinh(x)$ implies that $du = \cosh(x)dx$, and the bounds become

$$u(0) = \sinh(0) = 0$$
$$u(\ln(3)) = \sinh(\ln(3)) = \frac{1}{2}\left(e^{\ln(3)} - e^{-}\ln(3)\right) = \frac{1}{2}\left(3 - \frac{1}{3}\right) = \frac{4}{3}.$$

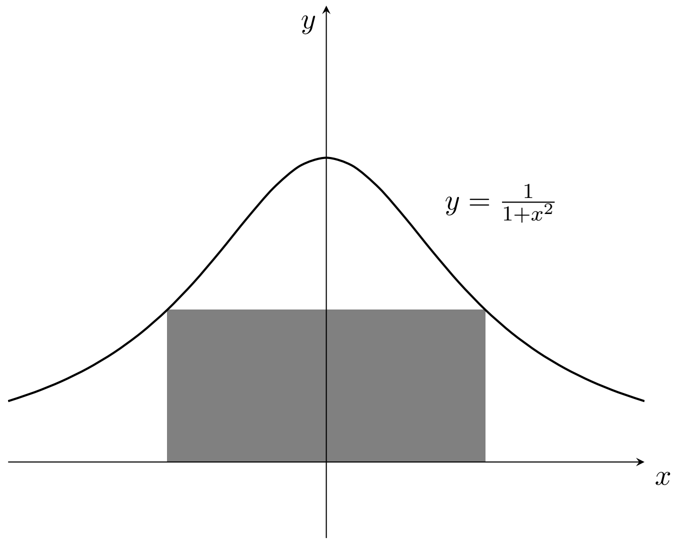
Evaluating the integral:

$$\begin{aligned}\int_0^{\ln(3)} \sinh(x)\cosh(x)\,dx &= \int_0^{4/3} u\,du \\ &= \frac{1}{2}u^2\Big|_0^{4/3} \\ &= \frac{1}{2}\left(\frac{4}{3}\right)^2 \\ &= \frac{8}{9}.\end{aligned}$$

(c) The limit yields the indeterminate form $0 \cdot (-\infty)$, so we apply L'Hôpital's rule:

$$\begin{aligned}\lim_{x\to 0^+} x^2\ln(x^2) &= \lim_{x\to 0^+} \frac{\ln(x^2)}{1/x^2} \\ &\overset{LH}{=} \lim_{x\to 0^+} \frac{2x/x^2}{-2/x^3} \\ &= \lim_{x\to 0^+} \left(\frac{2}{x}\right)\left(-\frac{x^3}{2}\right) \\ &= -\lim_{x\to 0^+} x^2 \\ &= 0.\end{aligned}$$

3. (16 points) Find the area of the largest rectangle which is symmetric around the y-axis, bounded below by the x-axis, and which has two corners touching the graph of $f(x) = \dfrac{1}{1+x^2}$. *Fully justify your answer by using an appropriate test.*



Solution:

Let x be the coordinate at the edge of the rectangle. Then the width is $2x$, and the height is $\dfrac{1}{1+x^2}$. Hence, the area is

$$A(x) = \frac{2x}{1+x^2}.$$

The derivative is given by

$$A'(x) = \frac{2(1+x^2) - (2x)(2x)}{(1+x^2)^2} = \frac{2-2x^2}{(1+x^2)^2},$$

which has domain $(-\infty, \infty)$. Hence, the critical values satisfy

$$0 = A'(x) \implies 0 = 2 - 2x^2 \implies x = \pm 1.$$

Since $x = -1$ yields a negative area, we use the first derivative test on $x = 1$.

$$A'(0) = \frac{2-2(0^2)}{(1+0^2)^2} > 0$$
$$A'(2) = \frac{2-2(2^2)}{(1+2^2)^2} < 0.$$

By the first derivative test, $x = 1$ yields a maximum area, which is

$$A(1) = \frac{2(1)}{1+1^2} = 1.$$

4. (18 points) A bug flying in a straight line starts decelerating at time $t = 0$ at a constant rate of 1 ft/s^2 for 5 seconds. Answer the following questions about the bug over the time interval $0 \le t \le 5$.

   (a) Find the bug's velocity as a function of time, given that its velocity at $t = 0$ is 2 ft/s.

   (b) What is the bug's displacement over the time interval $0 \le t \le 5$?

   (c) The bug changes direction at least once during the 5 seconds. What is the total distance the bug travels over the time interval $0 \le t \le 5$?

Solution:

(a) Since $a(t) = -1$ and $v(0) = 2$, integrating $a(t)$ yields $v(t) = -t + 2$.

(b) Integrating $v(t)$ and using the fact that $s(0) = 0$ yields $s(t) = -\frac{1}{2}t^2 + 2t$. Hence, the total displacement is $s(5) = -\frac{5}{2}$ feet.

(c) Note that the bug changes direction when $0 = v(t) = -t + 2$, so at $t = 2$. For $t < 2$, the velocity is positive, and for $t > 2$, the velocity is negative. Thus, the total distance D is given by

$$\begin{aligned} D &= \int_0^5 |v(t)|\,dt \\ &= \int_0^2 v(t)\,dt - \int_2^5 v(t)\,dt \\ &= (s(2) - s(0)) - (s(5) - s(2)) \\ &= 2s(2) - s(5) - s(0) \\ &= 4 - \left(-\frac{5}{2}\right) - 0 \\ &= \frac{13}{2}, \end{aligned}$$

so the bug travels a total of $13/2 = 6.5$ feet.

Hence, but

5. (12 points) For what value of a is the following function continuous?

$$f(x) = \begin{cases} 2x^2 - x + a, & x \le 0 \\ \dfrac{x}{2\sin(x)}, & x > 0 \end{cases}$$

Justify your answer with appropriate computations.

Solution:

For $f(x)$ to be continuous, we need

$$\lim_{x \to 0^-} f(x) = f(0) = \lim_{x \to 0^+} f(x).$$

Since $x^2 - x + a$ is a polynomial, it is continuous for any choice of a; hence, $f(x)$ is continuous at 0 from the left. We only need to choose a so that

$$a = f(0) = \lim_{x \to 0^+} \frac{x}{2\sin(x)} = \frac{1}{2}\left(\lim_{x \to 0^+} \frac{\sin(x)}{x}\right)^{-1} = \frac{1}{2}(1)^{-1} = \frac{1}{2}.$$

6. (18 points) Consider the function

$$g(x) = \arctan(x) + \frac{1}{x^2 - 4}$$

(a) Find the domain of the function, and give your answer in interval notation.

(b) Find all horizontal asymptotes of $g(x)$, and justify your answer with limits.

Solution:

(a) The domain of $\arctan(x)$ is $(-\infty, \infty)$, and the domain of $\dfrac{1}{x^2-4}$ is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$. Hence, the domain of $g(x)$ is also given by

$$(-\infty, -2) \cup (-2, 2) \cup (2, \infty).$$

(b)

$$\begin{aligned} \lim_{x \to \pm\infty} g(x) &= \lim_{x \to \pm\infty} \left(\arctan(x) + \frac{1}{x^2-4} \right) \\ &= \lim_{x \to \pm\infty} \arctan(x) + \underbrace{\lim_{x \to \pm\infty} \frac{1}{x^2-4}}_{=0} \\ &= \pm\frac{\pi}{2}. \end{aligned}$$

Therefore, $g(x)$ has horizontal asymptotes at $\frac{\pi}{2}$ (when $x \to \infty$) and $-\frac{\pi}{2}$ (when $x \to -\infty$).

7. (16 points) The half-life of the chemical element cobalt-56 is approximately 77 days. Suppose we have a 10 milligram sample of cobalt-56..

(a) Find a formula for the mass of cobalt-56 remaining after t days.

(b) How long will it take for only 1 milligram of cobalt-56 to remain in the sample? *It is OK for your answer to have a logarithm in it.*

Solution:

(a) Suppose that $m(t)$ is the mass of cobalt-56 remaining after t days. Using the law of natural decay, we know that

$$m(t) = 10e^{kt},$$

so we need to solve for the constant k. Using the information about the half-life we have

$$5 = 10e^{k(77)} \implies \frac{1}{2} = e^{77k} \implies \ln(1/2) = 77k \implies k = \frac{\ln(1/2)}{77}.$$

Hence, the following formulas for $m(t)$ are all valid:

$$\begin{aligned} m(t) &= 10e^{(\ln(1/2)/77)t} \\ &= 10\left(\frac{1}{2}\right)^{t/77} \\ &= 10e^{-(\ln(2)/77)t} \\ &= 10(2)^{-t/77}. \end{aligned}$$

(b) A single milligram of cobalt-56 will remain when

$$1 = 10\left(\frac{1}{2}\right)^{t/77} \implies \frac{\ln(1/10)}{\ln(1/2)} = \frac{t}{77} \implies t = \frac{-77\ln(10)}{-\ln(2)} \implies t = \frac{77\ln(10)}{\ln(2)}$$

This time is approximately 255.79 days.

8. (16 points) For each of the following questions, give a short justification for your answer.

(a) If $f(x)$ is an odd function and $\int_{-3}^{0} f(x)\,dx = \pi + 1$, find $\int_{-3}^{3} f(x)\,dx$.

(b) Find the absolute minimum of the function $f(x) = x \cdot 2^x$, if it exists.

(c) Evaluate the limit $\lim_{h\to 0} \dfrac{\arctan(3x+3h) - \arctan(3x)}{h}$.

(d) Suppose that $f(x)$ is differentiable everywhere, with $f(-1) = 1$ and $f(1) = 3$. Is there some value c such that $f'(c) = 1$?

Solution:

(a) The integral is 0. Since $f(x)$ is an odd function, it is a known fact that

$$\int_{-a}^{a} f(x)\,dx = 0,$$

which is true in particular for $a = 3$.

(b) The minimum value of $f(x)$ is given by

$$\left(-\frac{1}{\ln(2)}\right)\left(2^{-1/\ln(2)}.\right)$$

This is because

$$f'(x) = 2^x(1 + \ln(2)x) = 0$$

has a solution at $x = -1/\ln(2)$, which is the only critical value for $f(x)$. Also,

$$f''(x) = \ln(2)2^x(2 + \ln(2)x),$$

which is positive at $x = -1/\ln(2)$, which is thus a minimum for the function.

(c) The limit is $\dfrac{3}{1+9x^2}$. There are at least two ways to evaluate this limit. The first is to notice that the expression is the derivative of $y = \arctan(3x)$, which we know to be $\dfrac{3}{1+9x^2}$ using the chain rule and the derivative formula for $\arctan$.

Alternatively, the limit has the indeterminate form $\frac{0}{0}$, so we can use L'Hôpital's rule:

$$\lim_{h\to 0} \frac{\arctan(3x+3h) - \arctan(3x)}{h} \overset{LH}{=\!=} \lim_{h\to 0} \frac{\frac{3}{1+(3x+3h)^2} - 0}{1} = \frac{3}{1+9x^2}.$$

(d) Yes, there is such a c. Since $f(x)$ is differentiable everywhere, it is also continuous everywhere, and hence satisfies the Mean Value Theorem in particular on $[-1, 1]$. This means there is a $c \in (-1, 1)$ such that

$$f'(c) = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{3-1}{2} = 1.$$