1. (28 points) The following problems are not related.

   (a) (10 points) Evaluate the definite integral $\displaystyle\int_0^{\pi/2} \cos(x)\sqrt{1+2\sin(x)}\,dx$.

   (b) (10 points) Evaluate the definite integral $\displaystyle\int_{-1}^{2} \left|1-x^2\right|\,dx$.

   (c) (8 points) Suppose that $f(x) = \displaystyle\int_3^{\sqrt{x}} \frac{t^2+2}{t-1}\,dt$. Find $f'(4)$.

   Solution:

(a) Make the substitution $u = 1+2\sin(x)$, so that $\frac{du}{2} = \cos(x)\,dx$ and $u(0) = 1$, u$(\pi/2) = 3$. Then

$$\begin{aligned}\int_0^{\pi/2} \cos(x)\sqrt{1+2\sin(x)}\,dx &= \frac{1}{2}\int_1^3 \sqrt{u}\,du \\ &= \frac{1}{2}\left(\frac{2}{3}u^{3/2}\right)\Big|_1^3 \\ &= \frac{1}{3}(3^{3/2}-1^{3/2}) \\ &= \frac{1}{3}(3\sqrt{3}-1).\end{aligned}$$

(b) Note that $1-x^2 = -(x-1)(x+1)$, so $f(x) > 0$ on $[-1,1]$ and $f(x) < 0$ on $[1,2]$. Then

$$\begin{aligned}\int_{-1}^{2} |1-x^2|\,dx &= \int_{-1}^{1}(1-x^2)\,dx - \int_1^2 (1-x^2)\,dx \\ &= \left(x-\frac{1}{3}x^3\right)\Big|_{-1}^{1} - \left(x-\frac{1}{3}x^3\right)\Big|_1^2 \\ &= \left(\frac{2}{3}+\frac{2}{3}\right) - \left(-\frac{2}{3}-\frac{2}{3}\right) \\ &= \frac{8}{3}.\end{aligned}$$

(c)

$$f'(4) = \left(\frac{1}{2\sqrt{x}}\cdot\frac{x+2}{\sqrt{x}-1}\right)\Big|_{x=4} = \frac{1}{2\cdot 2}\cdot\frac{4+2}{2-1} = \frac{3}{2}.$$

2. (24 points) The following problems are not related.

   (a) (10 points) Approximate the area of the region bounded by the function $f(x) = 2\cos(x)+2$ and the x-axis on the interval $[-\pi/2, 3\pi/2]$ by using four approximating rectangles; take the sample points to be the right endpoints.

(b) (14 points) Evaluate the limit $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left(\frac{i^3}{n^3}+\frac{2i}{n}\right)$ using summation formulas, or by evaluating an appropriate definite integral.

Solution:

(a)

$$\begin{aligned}\int_{-\pi/2}^{3\pi/2} 2\cos(x)+2\ dx &\approx \left(\frac{3\pi/2+\pi/2}{4}\right)(f(0)+f(\pi/2)+f(\pi)+f(3\pi/2))\\ &= \frac{\pi}{2}(4+2+0+2)\\ &= 4\pi.\end{aligned}$$

(b) Using summation formulas, we have

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left(\frac{i^3}{n^3}+\frac{2i}{n}\right) &= \lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n} i^3+\lim_{n\to\infty}\frac{2}{n^2}\sum_{i=1}^{n} i\\ &= \lim_{n\to\infty}\frac{1}{n^4}\left(\frac{n(n+1)}{2}\right)^2+2\lim_{n\to\infty}\frac{1}{n^2}\left(\frac{n(n+1)}{2}\right)\\ &= \lim_{n\to\infty}\frac{n^4}{n^4}\left(\frac{(1)(1+1/n)}{2}\right)^2+2\lim_{n\to\infty}\frac{n^2}{n^2}\left(\frac{(1)(1+1/n)}{2}\right)\\ &= \frac{1}{4}\lim_{n\to\infty}(1+1/n)^2+\lim_{n\to\infty}(1+1/n)\\ &= \frac{1}{4}+1\\ &= \frac{5}{4}.\end{aligned}$$

Alternatively, one possible definite integral is given by

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left(\frac{i^3}{n^3}+\frac{2i}{n}\right)=\int_0^1 (x^3+2x)\,dx=\left(\frac{x^4}{4}+x^2\right)\Big|_0^1=\frac{1}{4}+1=\frac{5}{4}.$$

3. (16 points) The following problems are not related.

(a) (6 points) Suppose we want to approximate a solution to the equation $3x+2-\cos(x)=0$ using Newton's Method. What would the formula for x_{n+1} be? (To get full points for this question, you must provide the explicit formula for x_{n+1} in terms of x_n; the generic formula for Newton's Method is <u>not</u> sufficient.)

(b) (10 points) Suppose the acceleration of an object (in m/s^2) at any time t is given by $a(t)=6t^2-4$. Find the velocity $v(t)$ of the object at any time t, if $v(1)=2\,$m/s.

Solution:

(a) Letting $f(x) = 3x + 2 - \cos(x)$, we have that $f'(x) = 3 + \sin(x)$. Then

$$x_{n+1} = x_n - \frac{3x_n + 2 - \cos(x_n)}{3 + \sin(x_n)}.$$

(b) Since acceleration is the derivative of velocity,

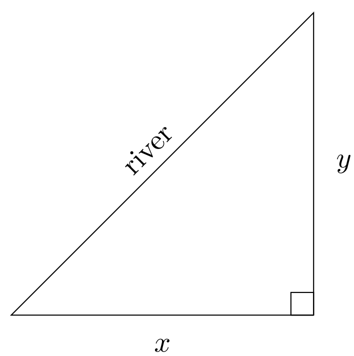
$$v(t) = \int (6t^2 - 4)\, dt = 2t^3 - 4t + C.$$

Using the fact that $v(1) = 2$, we have

$$2 = v(1) = 2(1)^3 - 4(1) + C \implies 2 = -2 + C \implies 4 = C.$$

Hence, $v(t) = 2t^3 - 4t + 4$.

4. (18 points) A farmer wants to fence off a small field in the shape of a right triangle. The hypotenuse of the triangle is along a riverbank, and the farmer will not need fencing there. If the farmer wants the area of the field to be $50\,\mathrm{m}^2$, what is the minimum amount of fencing they will need? Justify your answer with calculus techniques, and include appropriate units with your answer.



Solution:

We want to minimize the function $f = x + y$, subject to the constraint $50 = \frac{1}{2}xy$. Solving the constraint for y yields

$$y = \frac{100}{x}.$$

Plugging this in to the function f gives

$$f(x) = x + \frac{100}{x}.$$

We now solve for the critical numbers of f:

$$0 = f'(x) = 1 - \frac{100}{x^2} \implies x^2 = 100 \implies x = \pm 10 \implies x = 10.$$

Using the second derivative test,

$$f''(10) = \frac{200}{10^3} > 0,$$

so $x = 10$ yields a minimum amount of fencing for the field. Using the equation for y, we see that

$$y = \frac{100}{10} = 10,$$

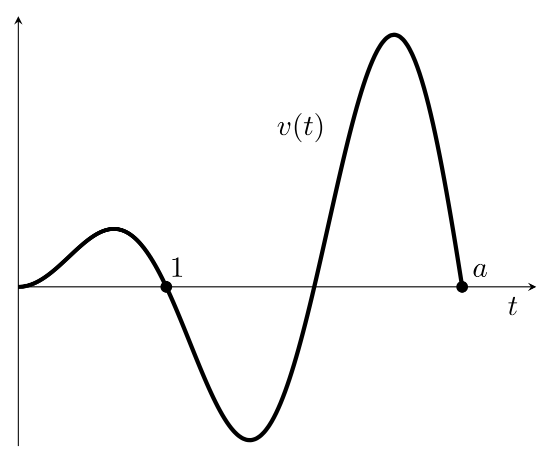
so $x = y = 10$. Hence, the farmer will need at minimum $10 + 10 = 20$ meters of fencing for the field.

5. (8 points) Write the expression $\int_{-1}^{2} f(x)\,dx + \int_{1}^{-1} f(x)\,dx + \int_{-3}^{1} f(x)\,dx$ as a single integral of the form $\int_{a}^{b} f(x)\,dx$.

Solution:

$$\begin{aligned}
\int_{-1}^{2} f(x)\,dx + \int_{1}^{-1} f(x)\,dx + \int_{-3}^{1} f(x)\,dx &= \int_{-1}^{2} f(x)\,dx - \int_{-1}^{1} f(x)\,dx + \int_{-3}^{1} f(x)\,dx \\
&= \int_{1}^{2} f(x)\,dx + \int_{-3}^{1} f(x)\,dx \\
&= \int_{-3}^{2} f(x)\,dx.
\end{aligned}$$

6. (6 points) Suppose the velocity $v(t)$ of a particle is given in the graph below:



Arrange the following quantities in order from smallest to largest:

(i) the total distance the particle travels from $t = 0$ to $t = a$

(ii) the displacement of the particle from $t = 0$ to $t = a$

(iii) the instantaneous acceleration of the particle at $t = 1$.

Note: no justification is required on this problem, but give your answer as a list of the numerals above. For example, $(i), (ii), (iii)$ would indicate that you believe item (i) is the smallest value, and item (iii) is the largest.

Solution:

(iii), (ii), (i)