1. (30 points) The following problems are not related.
(a) (10 points) Find the derivative of $g(x)=\sin \left(\frac{x^{2}+x}{3 x-1}\right)$. Do not simplify your answer.
(b) (14 points) Let $f(x)=\sqrt{4-x}$.
i. State the limit definition of the derivative for a function $f(x)$.
ii. Find $f^{\prime}(x)$ by using the definition of the derivative. You must use the limit definition to receive any credit.
(c) (6 points) If $f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{\sin (x+h)-\sin (x)}{h}$, find $f^{\prime}(\pi / 3)$.

## Solution:

(a)

$$
g^{\prime}(x)=\cos \left(\frac{x^{2}+x}{3 x+1}\right) \cdot\left(\frac{(3 x-1)(2 x+1)-\left(x^{2}+x\right)(3)}{(3 x-1)^{2}}\right) .
$$

(b) i. The derivative of a function $f(x)$ is defined to be

$$
f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h},
$$

so long as the limit exists.
ii. Using the limit definition:

$$
\begin{aligned}
f^{\prime}(x) & =\lim _{h \rightarrow 0} \frac{\sqrt{4-(x+h)}-\sqrt{4-x}}{h} \\
& =\lim _{h \rightarrow 0} \frac{\sqrt{4-(x+h)}-\sqrt{4-x}}{h} \cdot \frac{\sqrt{4-(x+h)}-\sqrt{4-x}}{\sqrt{4-(x+h)}-\sqrt{4-x}} \\
& =\lim _{h \rightarrow 0} \frac{4-(x+h)-4-x}{h(\sqrt{4-(x+h)}-\sqrt{4-x})} \\
& =\lim _{h \rightarrow 0} \frac{-h}{h(\sqrt{4-(x+h)}-\sqrt{4-x})} \\
& =\lim _{h \rightarrow 0} \frac{-1}{\sqrt{4-(x+h)}-\sqrt{4-x}} \\
& =-\frac{1}{2 \sqrt{4-x}} .
\end{aligned}
$$

(c) The right-hand side is the limit definition of the derivative for $\sin (x)$, so $f(x)=\sin (x)$. We know that $f^{\prime}(x)=\cos (x)$, so $f^{\prime}(\pi / 3)=\cos (\pi / 3)=1 / 2$.
2. (20 points) The following problems are not related.
(a) (8 points) The side length $h$ of a square is measured as 3 cm , with a maximum error of 0.1 cm . Use differentials to estimate:
i. the maximum error for the area of the square;
ii. the relative error for the area of the square.
(b) (12 points) You are flying a kite which has a constant height of 4 meters above the ground. The wind is carrying the kite horizontally away from you, and you have to let out string at a rate of 2 meters/minute. What is the horizontal speed of the kite when you have let out 5 meters of string?


## Solution:

(a) i. The area of a square is given by $A(h)=h^{2}$, so we have that

$$
d A=2 h d h=(2)(3)(0.1)=0.6
$$

Hence, the maximum error for the area of the square is $0.6 \mathrm{~cm}^{2}$ in this situation.
ii. For a side length measurement of 3 cm , the area is $9 \mathrm{~cm}^{2}$, so the relative error for the area is

$$
\frac{d A}{A}=\frac{0.6}{9}=\frac{3}{5} \cdot \frac{1}{9}=\frac{1}{15} \approx 6 . \overline{66} \%
$$

(b) Letting $z$ be the hypotenuse (the distance from you to the kite), and $x$ the horizontal distance, we know that

$$
z^{2}=x^{2}+4^{2} \Longrightarrow 2 z \frac{d z}{d t}=2 x \frac{d x}{d t} \Longrightarrow \frac{d x}{d t}=\frac{z}{x} \cdot \frac{d z}{d t}
$$

In order to get a value for $\frac{d x}{d t}$, we first need to get the value of $x$ when $z=5$ :

$$
5^{2}=x^{2}+4^{2} \Longrightarrow 25=x^{2}+16 \Longrightarrow 9=x^{2} \Longrightarrow x=3
$$

Hence, when $z=5$, we have that

$$
\frac{d x}{d t}=\frac{z}{x} \cdot \frac{d z}{d t}=\frac{5}{3} \cdot(2)=\frac{10}{3} \text { meters } / \mathrm{min} .
$$

3. (16 points) Consider the function $s(x)=-x^{3}+3 x+2$.
(a) Find the critical numbers of $s(x)$.
(b) Use the first derivative test to determine the points where $s(x)$ has a local maximum or local minimum. Give your answer as ordered pairs $(x, y)$.
(c) Find the absolute maximum and minimum values for the function $s(x)$ on the interval $[0,2]$.

## Solution:

(a) To find the critical numbers, first take the derivative

$$
s^{\prime}(x)=-3 x^{2}+3 .
$$

Since the domain of $s^{\prime}(x)$ is $(-\infty, \infty)$, the only critical numbers are solutions to the equation

$$
0=-3 x^{2}+3,
$$

and hence

$$
0=-3 x^{2}+3 \Longrightarrow 3 x^{2}=3 \Longrightarrow x^{2}=1 \Longrightarrow x= \pm 1
$$

So $x= \pm 1$ are the only critical numbers. The function values at the critical numbers are $s(-1)=0$, $s(1)=4$
(b) In order to determine whether each critical number $x= \pm 1$ is a local maximum, minimum, or neither, we apply the first derivative test to the intervals $(-\infty,-1),(-1,1)$, and $(1, \infty)$ by plugging the values $x=-2,0$, and 2 into $s^{\prime}(x)$.
Since $s^{\prime}(-2)=-9, s^{\prime}(0)=3$, and $s^{\prime}(2)=-9$, we know that $s(x)$ is decreasing on the intervals $(-\infty,-1)$ and $(1, \infty)$, and increasing on the interval $(-1,1)$. The first derivative test says that $(-1,0)$ is a local minimum, and $(1,4)$ is a local maximum.
(c) We compare the values $s(0)=2$ and $s(2)=0$ at the endpoints with the value at the only critical number in the interval, which is $s(1)=4$. Hence, 4 is the absolute maximum, and 0 is the absolute minimum on the interval.
4. (18 points) Suppose that $y$ is defined implicitly as a function of $x$ from the equation

$$
\cos (\pi y)=\frac{1}{2} x+y \cos (\pi x) .
$$

(a) Find the derivative $\frac{d y}{d x}$.
(b) Give an equation for the tangent line to this curve at the point where $y=0$.

## Solution:

(a) Implicitly differentiating gives

$$
-\pi \frac{d y}{d x} \sin (\pi y)=\frac{1}{2}+\frac{d y}{d x} \cos (\pi x)-\pi y \sin (\pi x) .
$$

Separating the terms with $\frac{d y}{d x}$ from those without it yields

$$
-\pi \frac{d y}{d x} \sin (\pi y)-\frac{d y}{d x} \cos (\pi x)=\frac{1}{2}-\pi y \sin (\pi x),
$$

then factoring out $\frac{d y}{d x}$ and solving for it gives

$$
\frac{d y}{d x}=-\frac{1 / 2-\pi y \sin (\pi x)}{\pi \sin (\pi y)+\cos (\pi x)} .
$$

(b) Plugging $y=0$ into the equation gives

$$
1=\frac{1}{2} x,
$$

which implies that $x=2$. Hence, we have to find the tangent line at the point $(2,0)$. Plugging these values into the formula for $\frac{d y}{d x}$, we find that

$$
\left.\frac{d y}{d x}\right|_{(x, y)=(2,0)}=-\frac{1 / 2-0}{0+1}=-\frac{1}{2}
$$

Then an equation for the tangent line to the curve at $(2,0)$ is given by

$$
y=-\frac{1}{2}(x-2)
$$

5. (16 points) Consider the function $f(x)=\frac{1}{x}$ on the interval $[2,4]$.
(a) (8 points) State the Mean Value Theorem and verify that $f(x)$ satisfies the hypotheses on the given interval.
(b) (8 points) Find all numbers $c$ that satisfy the conclusion of the Mean Value Theorem for $f(x)$ on the interval $[2,4]$.

## Solution:

(a) If $f(x)$ is continuous on $[a, b]$ and differentiable on $(a, b)$, then there is a $c$ in the interval $(a, b)$ such that

$$
f^{\prime}(c)=\frac{f(b)-f(a)}{b-a}
$$

The only value where $f(x)$ is discontinuous is $x=0$, so $f(x)$ is continuous on $[2,4]$. The function $f(x)$ is differentiable on $[2,4]$, since $f^{\prime}(x)=-\frac{1}{x^{2}}$, which is undefined only at $x=0$.
(b) The average rate of change of $f(x)$ on the interval $[2,4]$ is given by

$$
\frac{1 / 4-1 / 2}{4-2}=-\frac{1}{8}
$$

Since $f(x)$ satisfies the hypotheses of the Mean Value Theorem, there is at least one $c$ such that

$$
f^{\prime}(c)=-\frac{1}{c^{2}}=-\frac{1}{8} \Longrightarrow c^{2}=8 \Longrightarrow c= \pm 2 \sqrt{2}
$$

but the only value in the interval $[2,4]$ is $c=2 \sqrt{2}$.

