1. (30 points) The following problems are not related.

   (a) (10 points) Find the derivative of $g(x) = \sin\left(\frac{x^2+x}{3x-1}\right)$. Do not simplify your answer.

   (b) (14 points) Let $f(x) = \sqrt{4-x}$.

      i. State the limit definition of the derivative for a function $f(x)$.

      ii. Find $f'(x)$ by using the definition of the derivative. *You must use the limit definition to receive any credit.*

   (c) (6 points) If $f'(x) = \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h}$, find $f'(\pi/3)$.

   Solution:

(a)
$$g'(x) = \cos\left(\frac{x^2+x}{3x+1}\right) \cdot \left(\frac{(3x-1)(2x+1)-(x^2+x)(3)}{(3x-1)^2}\right).$$

(b) i. The derivative of a function $f(x)$ is defined to be

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h},$$

so long as the limit exists.

ii. Using the limit definition:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sqrt{4-(x+h)}-\sqrt{4-x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{4-(x+h)}-\sqrt{4-x}}{h} \cdot \frac{\sqrt{4-(x+h)}+\sqrt{4-x}}{\sqrt{4-(x+h)}+\sqrt{4-x}} \\
&= \lim_{h\to 0} \frac{4-(x+h)-4-x}{h(\sqrt{4-(x+h)}+\sqrt{4-x})} \\
&= \lim_{h\to 0} \frac{-h}{h(\sqrt{4-(x+h)}+\sqrt{4-x})} \\
&= \lim_{h\to 0} \frac{-1}{\sqrt{4-(x+h)}+\sqrt{4-x}} \\
&= -\frac{1}{2\sqrt{4-x}}.
\end{aligned}$$

(c) The right-hand side is the limit definition of the derivative for $\sin(x)$, so $f(x) = \sin(x)$. We know that $f'(x) = \cos(x)$, so $f'(\pi/3) = \cos(\pi/3) = 1/2$.

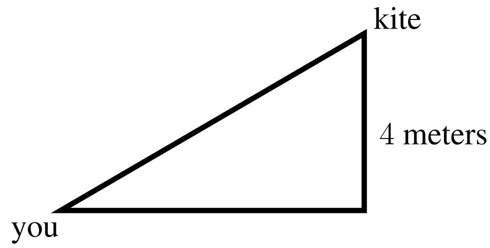
2. (20 points) The following problems are not related.

   (a) (8 points) The side length h of a square is measured as 3 cm, with a maximum error of 0.1 cm. Use differentials to estimate:

i. the maximum error for the area of the square;

ii. the relative error for the area of the square.

(b) (12 points) You are flying a kite which has a constant height of 4 meters above the ground. The wind is carrying the kite horizontally away from you, and you have to let out string at a rate of 2 meters/minute. What is the horizontal speed of the kite when you have let out 5 meters of string?



Solution:

(a) i. The area of a square is given by $A(h) = h^2$, so we have that

$$dA = 2h dh = (2)(3)(0.1) = 0.6.$$

Hence, the maximum error for the area of the square is 0.6 cm^2 in this situation.

ii. For a side length measurement of 3 cm, the area is 9 cm^2, so the relative error for the area is

$$\frac{dA}{A} = \frac{0.6}{9} = \frac{3}{5} \cdot \frac{1}{9} = \frac{1}{15} \approx 6.\overline{66}\%.$$

(b) Letting z be the hypotenuse (the distance from you to the kite), and x the horizontal distance, we know that

$$z^2 = x^2 + 4^2 \implies 2z\frac{dz}{dt} = 2x\frac{dx}{dt} \implies \frac{dx}{dt} = \frac{z}{x} \cdot \frac{dz}{dt}.$$

In order to get a value for $\frac{dx}{dt}$, we first need to get the value of x when $z = 5$:

$$5^2 = x^2 + 4^2 \implies 25 = x^2 + 16 \implies 9 = x^2 \implies x = 3.$$

Hence, when $z = 5$, we have that

$$\frac{dx}{dt} = \frac{z}{x} \cdot \frac{dz}{dt} = \frac{5}{3} \cdot (2) = \frac{10}{3} \text{ meters/min}.$$

3. (16 points) Consider the function $s(x) = -x^3 + 3x + 2$.

(a) Find the critical numbers of $s(x)$.

(b) Use the first derivative test to determine the points where $s(x)$ has a local maximum or local minimum. *Give your answer as ordered pairs* (x, y).

(c) Find the absolute maximum and minimum values for the function $s(x)$ on the interval $[0, 2]$.

Solution:

(a) To find the critical numbers, first take the derivative

$$s'(x) = -3x^2 + 3.$$

Since the domain of $s'(x)$ is $(-\infty, \infty)$, the only critical numbers are solutions to the equation

$$0 = -3x^2 + 3,$$

and hence

$$0 = -3x^2 + 3 \implies 3x^2 = 3 \implies x^2 = 1 \implies x = \pm 1.$$

So $x = \pm 1$ are the only critical numbers. The function values at the critical numbers are $s(-1) = 0$, $s(1) = 4$

(b) In order to determine whether each critical number $x = \pm 1$ is a local maximum, minimum, or neither, we apply the first derivative test to the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$ by plugging the values $x = -2$, 0, and 2 into $s'(x)$.

Since $s'(-2) = -9$, $s'(0) = 3$, and $s'(2) = -9$, we know that $s(x)$ is decreasing on the intervals $(-\infty, -1)$ and $(1, \infty)$, and increasing on the interval $(-1, 1)$. The first derivative test says that $(-1, 0)$ is a local minimum, and $(1, 4)$ is a local maximum.

(c) We compare the values $s(0) = 2$ and $s(2) = 0$ at the endpoints with the value at the only critical number in the interval, which is $s(1) = 4$. Hence, 4 is the absolute maximum, and 0 is the absolute minimum on the interval.

4. (18 points) Suppose that y is defined implicitly as a function of x from the equation

$$\cos(\pi y) = \frac{1}{2}x + y\cos(\pi x).$$

(a) Find the derivative $\frac{dy}{dx}$.

(b) Give an equation for the tangent line to this curve at the point where $y = 0$.

Solution:

(a) Implicitly differentiating gives

$$-\pi\frac{dy}{dx}\sin(\pi y) = \frac{1}{2} + \frac{dy}{dx}\cos(\pi x) - \pi y\sin(\pi x).$$

Separating the terms with $\frac{dy}{dx}$ from those without it yields

$$-\pi\frac{dy}{dx}\sin(\pi y) - \frac{dy}{dx}\cos(\pi x) = \frac{1}{2} - \pi y\sin(\pi x),$$

then factoring out $\frac{dy}{dx}$ and solving for it gives

$$\frac{dy}{dx} = -\frac{1/2 - \pi y\sin(\pi x)}{\pi\sin(\pi y) + \cos(\pi x)}.$$

(b) Plugging $y = 0$ into the equation gives

$$1 = \frac{1}{2}x,$$

which implies that $x = 2$. Hence, we have to find the tangent line at the point $(2, 0)$. Plugging these values into the formula for $\frac{dy}{dx}$, we find that

$$\left.\frac{dy}{dx}\right|_{(x,y)=(2,0)} = -\frac{1/2 - 0}{0 + 1} = -\frac{1}{2}.$$

Then an equation for the tangent line to the curve at $(2, 0)$ is given by

$$y = -\frac{1}{2}(x - 2).$$

5. (16 points) Consider the function $f(x) = \dfrac{1}{x}$ on the interval $[2, 4]$.

   (a) (8 points) State the Mean Value Theorem and verify that $f(x)$ satisfies the hypotheses on the given interval.

   (b) (8 points) Find all numbers c that satisfy the conclusion of the Mean Value Theorem for $f(x)$ on the interval $[2, 4]$.

   Solution:

(a) If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there is a c in the interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The only value where $f(x)$ is discontinuous is $x = 0$, so $f(x)$ is continuous on $[2, 4]$. The function $f(x)$ is differentiable on $[2, 4]$, since $f'(x) = -\frac{1}{x^2}$, which is undefined only at $x = 0$.

(b) The average rate of change of $f(x)$ on the interval $[2, 4]$ is given by

$$\frac{1/4 - 1/2}{4 - 2} = -\frac{1}{8}.$$

Since $f(x)$ satisfies the hypotheses of the Mean Value Theorem, there is at least one c such that

$$f'(c) = -\frac{1}{c^2} = -\frac{1}{8} \implies c^2 = 8 \implies c = \pm 2\sqrt{2},$$

but the only value in the interval $[2, 4]$ is $c = 2\sqrt{2}$.