1. (24 points) The following problems are not related. If a limit does not exist, you must say so. If you use a theorem, clearly state its name and show that its hypotheses are satisfied.
(Reminder: You may not use L'Hôpital's Rule or "Dominance of Powers" in any solutions on this exam.)

   (a) $\displaystyle\lim_{x\to 0} \frac{\sec x}{4x\cot 2x}$

   (b) $\displaystyle\lim_{x\to \infty} \frac{\sin^2 x}{x}$

   (c) $\displaystyle\lim_{x\to 1} \frac{x-1}{2-\sqrt{5-x^2}}$

Solution:

(a)

$$\begin{aligned}
\lim_{x\to 0} \frac{\sec x}{4x\cot 2x} &= \lim_{x\to 0} \frac{1/\cos x}{4x\cdot \frac{\cos 2x}{\sin 2x}} \\
&= \lim_{x\to 0} \frac{\sin 2x}{4x\cos x\cos 2x} \\
&= \lim_{x\to 0} \frac{2\sin x}{4x\cos 2x} \\
&= \frac{1}{2}\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\left(\frac{1}{\cos 2x}\right) \\
&= \frac{1}{2}
\end{aligned}$$

(b) Note that

$$0 \le \sin^2 x \le 1 \implies 0 \le \frac{\sin^2 x}{x} \le \frac{1}{x},$$

and

$$0 = \lim_{x\to\infty} 0 = \lim_{x\to\infty} \frac{1}{x}.$$

By the Squeeze Theorem, we conclude that

$$\lim_{x\to\infty} \frac{\sin^2 x}{x} = 0.$$

(c)

$$\begin{aligned}
\lim_{x\to 1} \frac{x-1}{2-\sqrt{5-x^2}} &= \lim_{x\to 1} \frac{(x-1)(2+\sqrt{5-x^2})}{(2-\sqrt{5-x^2})(2+\sqrt{5-x^2})} \\
&= \lim_{x\to 1} \frac{(x-1)(2+\sqrt{5-x^2})}{x^2-1} \\
&= \lim_{x\to 1} \frac{2+\sqrt{5-x^2}}{x+1} \\
&= \frac{2+2}{2} \\
&= 2.
\end{aligned}$$

2. (21 points) The following problems are unrelated.

   (a) Given that $\csc\theta = \sqrt{5}$ and $\pi/2 < \theta < \pi$, find the values of $\tan\theta$ and $\cos(2\theta)$.

   (b) Find all values of x in the interval $[0, \pi]$ that satisfy $\tan x \sec x = 4\sin x$.

   (c) A squirrel is up a tree, and it sees a peanut on the ground some distance away. If the straight-line distance between the peanut and the squirrel is 50 ft, and the angle between the straight-line and the tree is $\pi/6$ radians, how far down the tree and across the ground must the squirrel travel to reach the peanut? *Give your answer with appropriate units.*

Solution:

(a) Since $\csc\theta = \sqrt{5}$, we know that $\sin\theta = \frac{1}{\sqrt{5}}$. Thus, the angle θ is opposite a side of length 1 in a right triangle with hypotenuse $\sqrt{5}$. The adjacent side to θ has length $\sqrt{(\sqrt{5})^2 - 1^2} = 2$. Hence, $\tan\theta = -\frac{1}{2}$.
Using a double-angle identity for cosine, we know that

$$\cos 2\theta = 1 - 2\sin^2\theta = 1 - 2\left(1/\sqrt{5}\right)^2 = 1 - \frac{2}{5} = \frac{3}{5}.$$

(b) Note that

$$\begin{aligned} \tan x \sec x = 4\sin x &\implies \frac{\sin x}{\cos^2 x} = 4\sin x \\ &\implies \sin x - 4\sin x\cos^2 x = 0 \\ &\implies (\sin x)\left(1 - 4\cos^2 x\right) = 0. \end{aligned}$$

Hence, solutions to the equation are solutions of $\sin x = 0$, which are $x = 0$ and $x = \pi$, and solutions of $1 - 4\cos^2 x = 0$:

$$1 - 4\cos^2 x = 0 \implies \pm\frac{1}{2}\cos x,$$

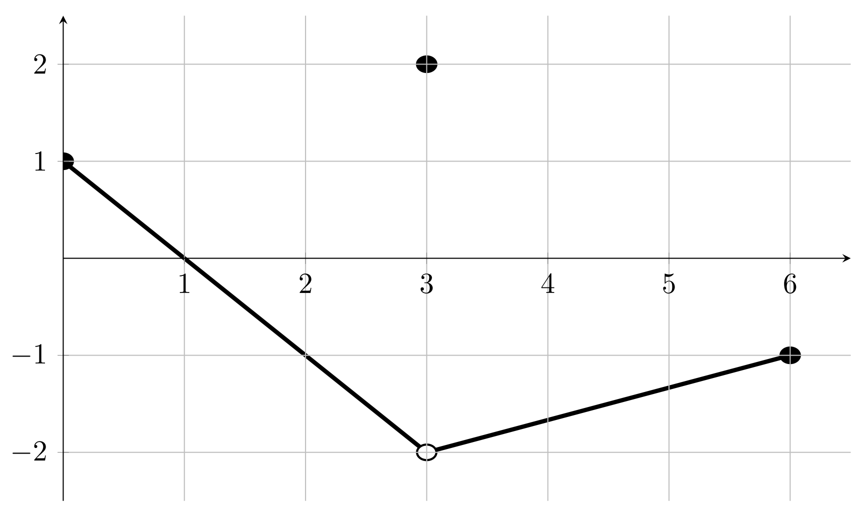
which has solutions $x = \pi/3$ and $x = 2\pi/3$. Therefore, the solutions to the given equation on the interval $[0, \pi]$ are

$$x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi.$$

(c) The distance down the tree is given by $y = 50\cos(\pi/6) = 25\sqrt{3}$, and the distance along the ground is $x = 50\sin(\pi/6) = 25$, so the total distance is given by

$$D = x + y = 25(1 + \sqrt{3}).$$

3. (15 points) Shown below is a graph of $y = f(x)$, which consists of two line segments with a single removable discontinuity.



(a) Find a formula for $f(x)$.

(b) Sketch a graph of $y = |f(x)| + 1$. Label the intercepts, if any.

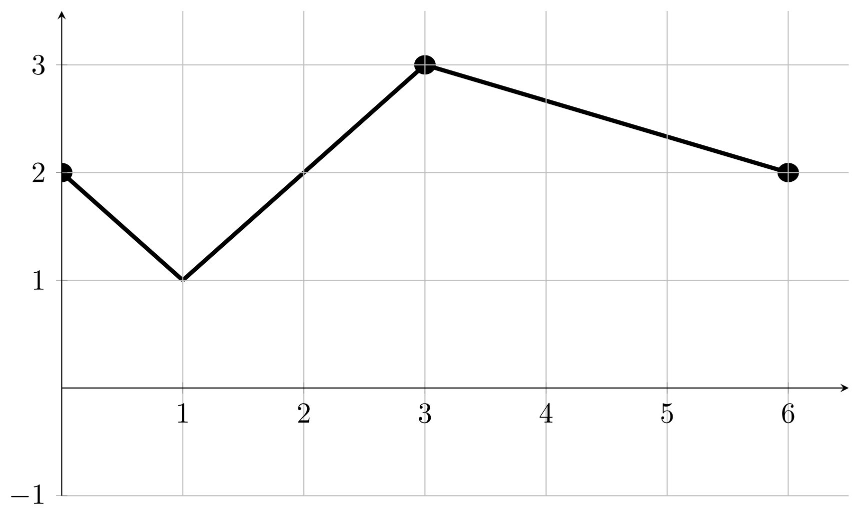
(c) Suppose we use the precise definition of a limit to verify the value of $\lim_{x\to 5} f(x)$, and we find that if $4 < x < 6$, then $-\frac{5}{3} < f(x) < -1$. What are the corresponding values of ϵ and δ? (recall the precise definition of a limit: *the limit of $f(x)$ as x approaches a is L if for every number $\epsilon > 0$, there is a corresponding $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.*

Solution:

(a)

$$f(x) = \begin{cases} 1 - x, & 0 \le x < 3 \\ 2, & x = 3 \\ \frac{1}{3}x - 3, & 3 < x \le 6 \end{cases}$$

(b)



(c) $\delta = 1$, $\epsilon = \frac{1}{3}$.

4. (20 points) Consider the function $g(x) = \dfrac{2x^2 - 12x + 16}{x^2 - 7x + 12}$.

(Reminder: You may not use L'Hôpital's Rule or "Dominance of Powers" in any solutions on this exam.)

(a) Find the domain of $g(x)$. Express your answer in interval notation.

(b) Find and classify all discontinuities of $g(x)$; justify your answers by calculating the appropriate limits.

(c) Find the horizontal asymptotes, if any; justify your answers by calculating the appropriate limits.

Solution:

(a) We can factor the numerator and denominator as

$$g(x) = \frac{2(x-2)(x-4)}{(x-3)(x-4)},$$

which shows that $g(x)$ is undefined at $x = 3$ and $x = 4$. Hence, the domain of $g(x)$ is $(-\infty, 3) \cup (3, 4) \cup (4, \infty)$.

(b) Note that we can cancel the $(x-4)$ factor in the numerator and denominator of $g(x)$, so

$$g(x) = \frac{2(x-2)}{x-3}$$

for all x except $x = 4$. Then

$$\lim_{x \to 4} g(x) = \lim_{x \to 4} \frac{2(x-2)}{x-3} = \frac{2(4-2)}{4-3} = 4,$$

which shows that $x = 4$ is a removable discontinuity for $g(x)$.
Also, $x = 3$ is an infinite discontinuity (or a vertical asymptote) for $g(x)$ because

$$\lim_{x \to 3^+} g(x) = \lim_{x \to 3^+} \frac{2(x-2)}{x-3} = \frac{2}{\underbrace{\lim_{x \to 3^+}(x-3)}_{\to 0^+}} = \infty.$$

(c)

$$\begin{aligned}
\lim_{x \to \infty} \frac{2x^2 - 12x + 16}{x^2 - 7x + 12} &= \lim_{x \to \infty} \frac{x^2(2 - 12/x - 16/x^2)}{x^2(1 - 7/x + 12/x^2)} \\
&= \lim_{x \to \infty} \frac{2 - 12/x - 16/x^2}{1 - 7/x + 12/x^2} \\
&= \frac{2 - 0 + 0}{1 - 0 + 0} \\
&= 2.
\end{aligned}$$

By a similar argument, $\lim_{x \to -\infty} g(x) = 2$.

5. (10 points) Consider the function

$$f(x) = \begin{cases} b\cos(\pi x), & x \le 1 \\ 3 - \sqrt{2x-2}, & x > 1 \end{cases}$$

Find the value of b such that $\lim_{x \to 1} f(x)$ exists. Justify your answer by calculating appropriate limits.

Solution: Note that

$$\begin{aligned}
\lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} (3 - \sqrt{2x-2}) \\
&= 3 - \sqrt{2(1) - 2} \\
&= 3,
\end{aligned}$$

So we need

$$3 = \lim_{x \to 1^-} b\cos(\pi x) = b\cos(\pi) = -b$$

for $\lim_{x \to 1} f(x)$ to exist. Hence, choosing $b = -3$ guarantees that the two-sided limit of $f(x)$ at exists, in which case it equals 3.

6. (10 points) Show that the equation $x - 2 = \sin x \cos x$ has at least one real solution. Indicate the interval where a solution can be found.

Solution: Let $f(x) = x - 2 - \sin x \cos x$. Then the given equation has a solution where $f(x) = 0$. Note that $f(x)$ is continuous because $\sin x \cos x$ is the product of continuous functions, which is continuous, and $x - 2$ is continuous because it's a polynomial. Then $f(x)$ is given by the difference of two continuous functions, and hence is continuous itself.

Also, $f(0) = -2 < 0$, and $f(\pi) = \pi - 2 > 0$. Since f is continuous everywhere, in particular on $[0, 2]$, the Intermediate Value Theorem guarantees that $f(x) = 0$ has a solution in the interval $(0, 2)$, and the given equation does as well.