1. (30 pts) The following three problems are not related.

   (a) Find the derivative of $y = \cos(x \sin x)$. Leave your answer unsimplified.

   (b) Find an equation for the line tangent to $y = \left(x^2+1\right)\left(2x^3+3\right)$ at $x = 1$. Express your answer in point-slope form.

   (c) Find dy/dx given
   $$\frac{y}{y^3+5} = 2x^{3/2} - 3$$
   by applying the quotient rule to the left side of the equation. Express your answer in terms of x and y.

Solution:

(a)

$$\begin{aligned} y &= \cos(x \sin x) \\ y' &= \boxed{-\sin(x \sin x)(x \cos x + \sin x)} \end{aligned}$$

(b) Differentiate the function using the product rule.

$$\begin{aligned} y &= \left(x^2+1\right)\left(2x^3+3\right) \\ y' &= \left(x^2+1\right)\left(6x^2\right) + \left(2x^3+3\right)(2x) \end{aligned}$$

Find the values of $y(1)$ and $y'(1)$.

$$\begin{aligned} y(1) &= 2 \cdot 5 = 10 \\ y'(1) &= 2(6) + 5(2) = 22 \end{aligned}$$

The tangent line in point-slope form at $x = 1$ is therefore

$$\boxed{y - 10 = 22(x-1)} \quad \text{or}$$
$$\boxed{y = 10 + 22(x-1)}.$$

(c)

$$\begin{aligned} \frac{y}{y^3+5} &= 2x^{3/2} - 3 \\ \frac{\left(y^3+5\right)\frac{dy}{dx} - y\left(3y^2\right)\frac{dy}{dx}}{(y^3+5)^2} &= 3\sqrt{x} \\ \frac{dy}{dx}\left(y^3+5-3y^3\right) &= 3\sqrt{x}\left(y^3+5\right)^2 \\ \frac{dy}{dx} &= \boxed{\frac{3\sqrt{x}\left(y^3+5\right)^2}{5-2y^3}} \end{aligned}$$

2. (18 pts)

   (a) State the limit definition of the derivative for a function $f(x)$.

   (b) Let $f(x) = 3x^2 + 7$.

      i. Use the limit definition to find $f'(x)$.

      ii. For the given $f(x)$, find all values of c that satisfy the conclusion of the Mean Value Theorem on the interval $[-1, 2]$. You may assume that $f(x)$ satisfies the hypotheses of the theorem.

Solution:

(a)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

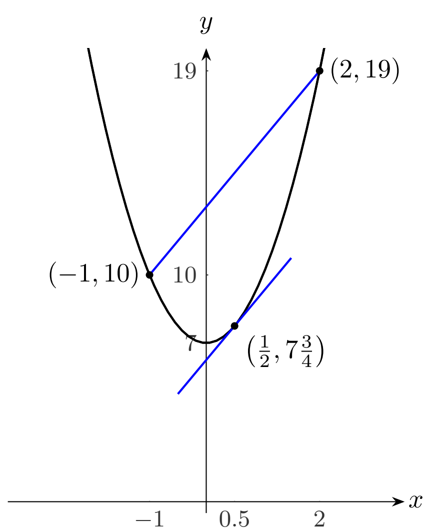
(b) $f(x) = 3x^2 + 7$

i.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{3(x+h)^2 + \cancel{7} - \left(3x^2 + \cancel{7}\right)}{h} \\ &= \lim_{h \to 0} \frac{3\left(x^2 + 2hx + h^2\right) - 3x^2}{h} \\ &= \lim_{h \to 0} \frac{\cancel{3x^2} + 6hx + 3h^2 - \cancel{3x^2}}{h} \\ &= \lim_{h \to 0} \frac{\cancel{h}(6x + 3h)}{\cancel{h}} \\ &= \boxed{6x} \end{aligned}$$

ii.

$$\begin{aligned} f'(c) &= \frac{f(b) - f(a)}{b - a} \\ 6c &= \frac{f(2) - f(-1)}{2 - (-1)} = \frac{19 - 10}{3} = \frac{9}{3} = 3 \\ c &= \boxed{\frac{1}{2}} \end{aligned}$$



3. (14 pts) Let $g(x) = 2 + \sqrt[3]{x}$.

   (a) Find the linearization of $g(x)$ at $x = 1$.

   (b) Use the linearization to approximate the value of $g(0.7)$.

   (c) Is the approximation an underestimate or overestimate? Be sure to justify your answer.

Solution:

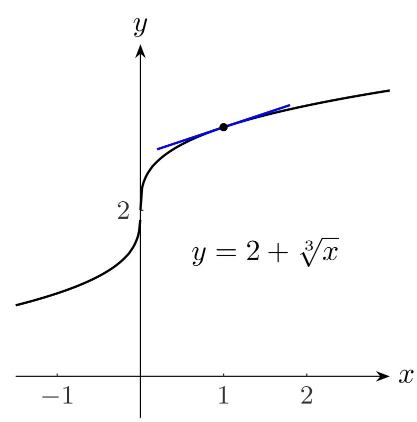
(a) The linearization of $g(x)$ at $a = 1$ is $L(x) = g(1) + g'(1)(x - 1)$.

$$g(x) = 2 + \sqrt[3]{x} \qquad\qquad g(1) = 2 + 1 = 3$$
$$g'(x) = \frac{1}{3}x^{-2/3} \qquad\qquad g'(1) = \frac{1}{3}$$

$$\begin{aligned} L(x) &= g(1) + g'(1)(x - 1) \\ &= \boxed{3 + \frac{1}{3}(x - 1)} = \frac{1}{3}x + \frac{8}{3} \end{aligned}$$

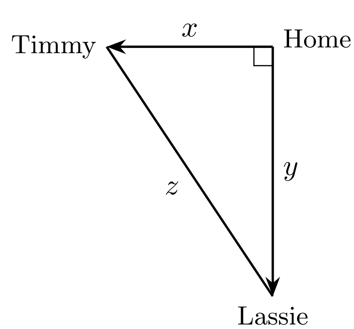
(b) $g(0.7) \approx L(0.7) = 3 + \frac{1}{3}(0.7 - 1) = 3 + \frac{1}{3}(-0.3) = 3 - 0.1 = \boxed{2.9}$

(c) The approximation is an $\boxed{\text{overestimate}}$ because the tangent line at $x = 1$ lies above the curve. (The actual value of $g(0.7)$ is about 2.8879.)



4. (12 pts) Timmy leaves home at 8 a.m. and walks west toward school at $\frac{3}{2}$ m/sec. At the same time, his dog, Lassie, heads south to visit the dog park, trotting at 2 m/sec. After 8 seconds, how fast is the distance between them changing?

Solution:



Let x and y represent Timmy's and Lassie's distances from home, respectively. Let z represent the distance between them. After 8 seconds, Timmy is $x = 8 \cdot \frac{3}{2} = 12$ meters from home, and Lassie is $y = 8 \cdot 2 = 16$ meters from home. By the Pythagorean Theorem, at that moment, the distance between them is $z = \sqrt{12^2 + 16^2} = \sqrt{400} = 20$ meters. We wish to find dz/dt.

$$\begin{aligned} z^2 &= x^2 + y^2 \\ 2z\frac{dz}{dt} &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\ z\frac{dz}{dt} &= x\frac{dx}{dt} + y\frac{dy}{dt} \\ 20\frac{dz}{dt} &= 12 \cdot \frac{3}{2} + 16 \cdot 2 \\ \frac{dz}{dt} &= \frac{18+32}{20} = \boxed{\frac{5}{2}\text{ m/sec}} \end{aligned}$$

5. (26 pts) Problems 5a and 5b are not related.

   (a) Let $h(x) = \left(x^2 - 4\right)^3$.

      i. Find the critical numbers of h.

      ii. Use the First Derivative Test to classify each critical number as a local minimum, local maximum, or neither.

   Solution:

      i. To find the critical numbers of $h(x) = \left(x^2 - 4\right)^3$, first differentiate the function.

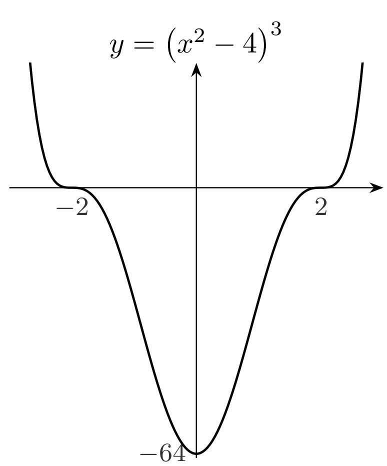
$$\begin{aligned} h'(x) &= 3\left(x^2 - 4\right)^2 (2x) \\ h'(x) &= 6x\left(x^2 - 4\right)^2 \end{aligned}$$

      The derivative is defined for all x. The critical numbers where $h'(x) = 0$ are $x = \boxed{0, \pm 2}$.

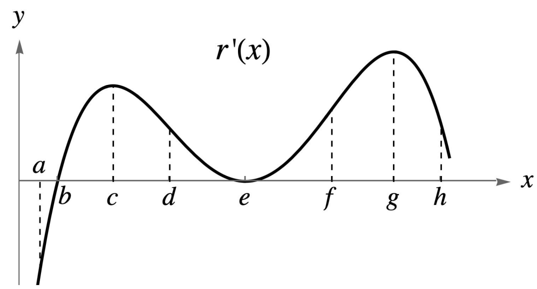
      ii. Note that the $\left(x^2 - 4\right)^2$ factor in the derivative is positive for all $x \neq \pm 2$, so the sign of $h'(x)$ is determined by the sign of $6x$.

Interval	$h'(x) = 6x\left(x^2 - 4\right)^2$	$h(x)$
$x < -2$	$-$	decreasing
$-2 < x < 0$	$-$	decreasing
$0 < x < 2$	$+$	increasing
$x > 2$	$+$	increasing

      Therefore there is a $\boxed{\text{local minimum at } x = 0}$. There are $\boxed{\text{no local extrema at } x = \pm 2}$.



(b) The graph of derivative $r'(x)$ is shown below.



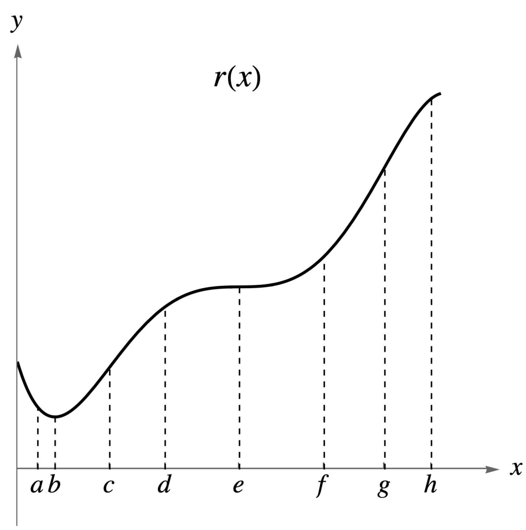
i. Among the x-coordinates a, b, c, d, e, f, g, and h, enter the coordinate(s) that correspond to each of the following quantities.

Solution: Because the derivative $r'(x)$ is negative on (a, b) and positive on (b, h), the function $r(x)$ is decreasing on (a, b) and increasing on (b, h). Therefore $r(x)$ has an absolute minimum at $x = b$ and an absolute maximum at $x = h$.

From the graph, $r'(x)$ has minimum and maximum values at $x = a$ and $x = g$, respectively.

The second derivative $r''(x)$ is minimized where the slope of $r'(x)$ is least at $x = h$. The second derivative is maximized where the slope of $r'(x)$ is greatest at $x = a$.

min $r(x)$ at $x = b$	min $r'(x)$ at $x = a$	min $r''(x)$ at $x = h$
max $r(x)$ at $x = h$	max $r'(x)$ at $x = g$	max $r''(x)$ at $x = a$



ii. At what value(s) of x does the function $r(x)$ have horizontal tangents?

Solution: The function $r(x)$ has horizontal tangents where $r'(x) = 0$.

$r(x)$ has horizontal tangent(s) at $x = b, e$

iii. On what interval(s) is the function $r(x)$ concave up or down? Use the given coordinates as endpoints of the intervals.

Solution: The function $r(x)$ is concave up where $r'(x)$ is increasing and $r''(x)$ is positive. It is concave down where $r'(x)$ is decreasing and $r''(x)$ is negative.

$r(x)$ is concave up on $(a, c), (e, g)$
$r(x)$ is concave down on $(c, e), (g, h)$