- 1. (30 pts) The following three problems are not related.
 - (a) Find the derivative of $y = \cos(x \sin x)$. Leave your answer unsimplified.
 - (b) Find an equation for the line tangent to $y = (x^2 + 1)(2x^3 + 3)$ at x = 1. Express your answer in point-slope form.
 - (c) Find dy/dx given

$$\frac{y}{y^3+5} = 2x^{3/2} - 3$$

by applying the quotient rule to the left side of the equation. Express your answer in terms of x and y.

Solution:

(a)

$$y = \cos(x \sin x)$$
$$y' = \boxed{-\sin(x \sin x)(x \cos x + \sin x)}$$

(b) Differentiate the function using the product rule.

$$y = (x^{2} + 1) (2x^{3} + 3)$$

$$y' = (x^{2} + 1) (6x^{2}) + (2x^{3} + 3) (2x)$$

Find the values of y(1) and y'(1).

$$y(1) = 2 \cdot 5 = 10$$

 $y'(1) = 2(6) + 5(2) = 22$

The tangent line in point-slope form at x = 1 is therefore

$$y - 10 = 22(x - 1)$$
 or
 $y = 10 + 22(x - 1)$.

(c)

$$\frac{y}{y^3 + 5} = 2x^{3/2} - 3$$
$$\frac{(y^3 + 5)\frac{dy}{dx} - y(3y^2)\frac{dy}{dx}}{(y^3 + 5)^2} = 3\sqrt{x}$$
$$\frac{dy}{dx}(y^3 + 5 - 3y^3) = 3\sqrt{x}(y^3 + 5)^2$$
$$\frac{dy}{dx} = \boxed{\frac{3\sqrt{x}(y^3 + 5)^2}{5 - 2y^3}}$$

2. (18 pts)

- (a) State the limit definition of the derivative for a function f(x).
- (b) Let $f(x) = 3x^2 + 7$.
 - i. Use the limit definition to find f'(x).
 - ii. For the given f(x), find all values of c that satisfy the conclusion of the Mean Value Theorem on the interval [-1, 2]. You may assume that f(x) satisfies the hypotheses of the theorem.

Solution:

(a)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(b) $f(x) = 3x^2 + 7$ i. $f'(x) = \lim_{h \to 0} \frac{3(x+h)^2 + 7 - (3x^2 + 7)}{h}$ $= \lim_{h \to 0} \frac{3(x^2 + 2hx + h^2) - 3x^2}{h}$ $= \lim_{h \to 0} \frac{3x^2 + 6hx + 3h^2 - 3x^2}{h}$ $= \lim_{h \to 0} \frac{\mathcal{K}(6x + 3h)}{\mathcal{K}}$ = [6x]

ii.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$6c = \frac{f(2) - f(-1)}{2 - (-1)} = \frac{19 - 10}{3} = \frac{9}{3} = 3$$

$$c = \boxed{\frac{1}{2}}$$

$$y$$

$$19$$

$$(2, 19)$$



- 3. (14 pts) Let $g(x) = 2 + \sqrt[3]{x}$.
 - (a) Find the linearization of g(x) at x = 1.
 - (b) Use the linearization to approximate the value of g(0.7).
 - (c) Is the approximation an underestimate or overestimate? Be sure to justify your answer.

Solution:

(a) The linearization of g(x) at a = 1 is L(x) = g(1) + g'(1)(x - 1).

$$g(x) = 2 + \sqrt[3]{x} \qquad g(1) = 2 + 1 = 3$$
$$g'(x) = \frac{1}{3}x^{-2/3} \qquad g'(1) = \frac{1}{3}$$
$$L(x) = g(1) + g'(1)(x - 1)$$
$$= \boxed{3 + \frac{1}{3}(x - 1)} = \frac{1}{3}x + \frac{8}{3}$$
(b) $g(0.7) \approx L(0.7) = 3 + \frac{1}{3}(0.7 - 1) = 3 + \frac{1}{3}(-0.3) = 3 - 0.1 = \boxed{2.9}$

(c) The approximation is an overestimate because the tangent line at x = 1 lies above the curve. (The actual value of g(0.7) is about 2.8879.)



4. (12 pts) Timmy leaves home at 8 a.m. and walks west toward school at $\frac{3}{2}$ m/sec. At the same time, his dog, Lassie, heads south to visit the dog park, trotting at 2 m/sec. After 8 seconds, how fast is the distance between them changing?

Solution:



Let x and y represent Timmy's and Lassie's distances from home, respectively. Let z represent the distance between them. After 8 seconds, Timmy is $x = 8 \cdot \frac{3}{2} = 12$ meters from home, and Lassie is $y = 8 \cdot 2 = 16$ meters from home. By the Pythagorean Theorem, at that moment, the distance between them is $z = \sqrt{12^2 + 16^2} = \sqrt{400} = 20$ meters. We wish to find dz/dt.

$$z^{2} = x^{2} + y^{2}$$

$$2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$z \frac{dz}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt}$$

$$20 \frac{dz}{dt} = 12 \cdot \frac{3}{2} + 16 \cdot 2$$

$$\frac{dz}{dt} = \frac{18 + 32}{20} = \boxed{\frac{5}{2}} \text{ m/sec}$$

- 5. (26 pts) Problems 5a and 5b are not related.
 - (a) Let $h(x) = (x^2 4)^3$.
 - i. Find the critical numbers of h.
 - ii. Use the First Derivative Test to classify each critical number as a local minimum, local maximum, or neither.

Solution:

i. To find the critical numbers of $h(x) = (x^2 - 4)^3$, first differentiate the function.

$$h'(x) = 3(x^{2} - 4)^{2}(2x)$$
$$h'(x) = 6x(x^{2} - 4)^{2}$$

The derivative is defined for all x. The critical numbers where h'(x) = 0 are $x = 0, \pm 2$.

ii. Note that the $(x^2 - 4)^2$ factor in the derivative is positive for all $x \neq \pm 2$, so the sign of h'(x) is determined by the sign of 6x.

Interval	$h'(x) = 6x \left(x^2 - 4\right)^2$	h(x)
x < -2	—	decreasing
-2 < x < 0	—	decreasing
0 < x < 2	+	increasing
x > 2	+	increasing

Therefore there is a local minimum at x = 0. There are no local extrema at $x = \pm 2$.



(b) The graph of derivative r'(x) is shown below.



i. Among the *x*-coordinates *a*, *b*, *c*, *d*, *e*, *f*, *g*, and *h*, enter the coordinate(s) that correspond to each of the following quantities.

min $r(x)$ at $x =$	min $r'(x)$ at $x =$	min $r''(x)$ at $x =$
$\max r(x)$ at $x =$	$\max r'(x)$ at $x =$	$\max r''(x) \text{ at } x =$

Solution: Because the derivative r'(x) is negative on (a, b) and positive on (b, h), the function r(x) is decreasing on (a, b) and increasing on (b, h). Therefore r(x) has an absolute minimum at x = b and an absolute maximum at x = h.

From the graph, r'(x) has minimum and maximum values at x = a and x = g, respectively.

The second derivative r''(x) is minimized where the slope of r'(x) is least at x = h. The second derivative is maximized where the slope of r'(x) is greatest at x = a.

$\min r(x) \text{ at } x = b$	min $r'(x)$ at $x = a$	min $r''(x)$ at $x = h$
$\max r(x) \text{ at } x = h$	$\max r'(x) \text{ at } x = g$	$\max r''(x) \text{ at } x = a$



ii. At what value(s) of x does the function r(x) have horizontal tangents?

Solution: The function r(x) has horizontal tangents where r'(x) = 0.

r(x) has horizontal tangent(s) at x = b, e

iii. On what interval(s) is the function r(x) concave up or down? Use the given coordinates as endpoints of the intervals.

Solution: The function r(x) is concave up where r'(x) is increasing and r''(x) is positive. It is concave down where r'(x) is decreasing and r''(x) is negative.

r(x) is concave up on (a, c), (e, g)r(x) is concave down on (c, e), (g, h)