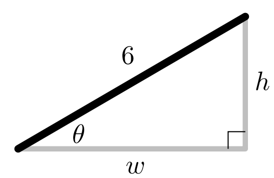
1. (22 pts)

   (a) A 6-meter ramp forms an angle of θ with the ground, $0 < \theta < \frac{\pi}{2}$. The ramp has a horizontal width of w meters and a vertical height of h meters, as shown in the diagram.



   i. If $\csc\theta = 6$, what is the value of w?
   ii. For what value(s) of θ will $w = \frac{h}{\sqrt{3}}$?

   (b) Find all values of θ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ that satisfy $\sec\theta\tan\theta = 2\sin\theta$.

Solution:

(a) i.

$$\csc\theta = \frac{1}{\sin\theta} = \frac{6}{h} = 6 \implies h = 1 \implies w = \sqrt{6^2 - 1^2} = \boxed{\sqrt{35}}.$$

ii. Because $\tan\theta = \dfrac{h}{w}$,

$$w = \frac{h}{\sqrt{3}} \implies \frac{h}{w} = \tan\theta = \sqrt{3} \implies \theta = \boxed{\frac{\pi}{3}}.$$

(b)

$$\begin{aligned}
\sec\theta\tan\theta &= 2\sin\theta \\
\frac{\sin\theta}{\cos^2\theta} &= 2\sin\theta \\
\frac{\sin\theta}{\cos^2\theta} - 2\sin\theta &= 0 \\
(\sin\theta)\left(\frac{1}{\cos^2\theta} - 2\right) &= 0
\end{aligned}$$

Then either $\sin\theta = 0 \implies \theta = \boxed{0}$, or

$$\frac{1}{\cos^2\theta} - 2 = 0 \implies \frac{1}{\cos^2\theta} = 2 \implies \cos^2\theta = \frac{1}{2} \implies \cos\theta = \pm\frac{1}{\sqrt{2}} \implies \theta = \boxed{\pm\frac{\pi}{4}}$$

in the given interval $(-\pi/2, \pi/2)$.

2. (18 pts) A cyclist covered 12 miles in 30 minutes, traveling at a constant rate. Over the next 20 minutes, the cyclist traveled at a slower constant rate and covered 6 miles.

   (a) Write a formula for $d(t)$, the distance in miles traveled by the cyclist as a function of time t in minutes, $0 \le t \le 50$. Simplify your answer.

   (b) Find the cyclist's average speed (in miles per minute) over the interval $t = 20$ to 40 minutes.

Solution:

(a) For the first 30 minutes, the speed of the cyclist was $\frac{12}{30} = \frac{2}{5}$ mi/min. For the next 20 minutes, the cyclist's speed was $\frac{6}{20} = \frac{3}{10}$ mi/min. Using the point $(30, 12)$ and the point-slope form of a linear function gives

$$12 + \frac{3}{10}(t - 30) = \frac{3}{10}t + 3.$$

A formula for $d(t)$ is

$$d(t) = \begin{cases} \dfrac{2}{5}t, & 0 \le t \le 30 \\ \dfrac{3}{10}t + 3, & 30 \le t \le 50. \end{cases}$$

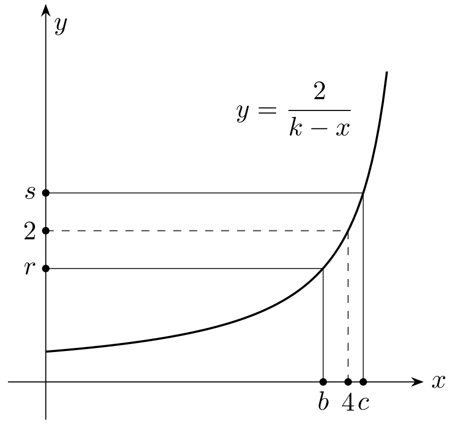
(b) The average speed from $t = 20$ to 40 minutes equals the distance traveled divided by the elapsed time:

$$\frac{\Delta d}{\Delta t} = \frac{d(40) - d(20)}{40 - 20}.$$

Because $d(40) = \frac{3}{10} \cdot 40 + 3 = 15$ miles and $d(20) = \frac{2}{5} \cdot 20 = 8$ miles, the average speed was

$$\frac{d(40) - d(20)}{40 - 20} = \frac{15 - 8}{20} = \boxed{\frac{7}{20} \text{ mi/min}}.$$

3. (12 pts)



The graph above uses the precise definition of a limit to illustrate $\lim\limits_{x \to 4} \left(\dfrac{2}{k-x} \right) = 2$ for some constant k with $\epsilon = \frac{1}{2}$. Find the values of k, r, s, b, c, and a corresponding δ.

Solution:

No justification is necessary.

$k = 5$	$r = \frac{3}{2}$	$s = \frac{5}{2}$
$b = \frac{11}{3}$	$c = \frac{21}{5}$	$\delta = \frac{1}{5}$

To find k, solve $y(4) = 2 \implies \dfrac{2}{k-4} = 2 \implies k - 4 = 1 \implies k = \boxed{5}$.

Because $\epsilon = \dfrac{1}{2}$, the value of r is $2 - \epsilon = \boxed{\dfrac{3}{2}}$ and the value of s is $2 + \epsilon = \boxed{\dfrac{5}{2}}$.

To find b, solve $y(b) = r \implies \dfrac{2}{5-b} = \dfrac{3}{2} \implies 15 - 3b = 4 \implies b = \boxed{\dfrac{11}{3}}$.

To find c, solve $y(c) = s \implies \dfrac{2}{5-c} = \dfrac{5}{2} \implies 25 - 5c = 4 \implies c = \boxed{\dfrac{21}{5}}$.

Because c is closer to 4 than b, choose $\delta = c - 4 = \boxed{\dfrac{1}{5}}$.

4. (25 pts) Evaluate the following limits. Fully simplify your answers. (*Reminder:* You may not use L'Hopital's Rule or "dominance of powers" arguments.)

(a) $\lim_{u \to 13} \frac{u - 13}{\sqrt{u} - \sqrt{13}}$

(b) $\lim_{\theta \to 0} \frac{7\theta \cot(3\theta)}{\sec(5\theta)}$

(c) $\lim_{x \to -3} \frac{3 - |x|}{9 - x^2}$

Solution:

(a)

$$\lim_{u \to 13} \frac{u - 13}{\sqrt{u} - \sqrt{13}} \cdot \frac{\sqrt{u} + \sqrt{13}}{\sqrt{u} + \sqrt{13}} = \lim_{u \to 13} \frac{\cancel{(u - 13)}\left(\sqrt{u} + \sqrt{13}\right)}{\cancel{u - 13}} = \boxed{2\sqrt{13}}$$

(b) Use the theorem $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{x}{\sin x} = 1$.

$$\begin{aligned} \lim_{\theta \to 0} \frac{7\theta \cot(3\theta)}{\sec(5\theta)} &= \lim_{\theta \to 0} \frac{7\theta}{1} \cdot \frac{\cos(3\theta)}{\sin(3\theta)} \cdot \frac{\cos(5\theta)}{1} \\ &= \lim_{\theta \to 0} \frac{7\theta}{3\theta} \cdot \frac{3\theta}{\sin(3\theta)} \cdot \frac{\cos(3\theta)}{1} \cdot \frac{\cos(5\theta)}{1} \\ &= \frac{7}{3} \cdot 1 \cdot 1 \cdot 1 = \boxed{\frac{7}{3}} \end{aligned}$$

(c) Substitute $|x| = -x$ for $x < 0$.

$$\lim_{x \to -3} \frac{3 - |x|}{9 - x^2} = \lim_{x \to -3} \frac{3 - (-x)}{9 - x^2} = \lim_{x \to -3} \frac{\cancel{3 + x}}{(3 - x)\cancel{(3 + x)}} = \boxed{\frac{1}{6}}$$

5. (23 pts) The following two problems are not related.

(a) Let $f(x) = \dfrac{5x+15}{x^2-4x-21}$. Determine all values of x at which $f(x)$ is discontinuous. For each value, identify the type of discontinuity. Justify using appropriate limits.

(b) Prove that $P(x) = 3x^3 - 4x^2 - 3x + 1$ has at least **two real roots** between $x = 0$ and $x = 2$.

Solution:

(a) The rational function $f(x)$ is continuous in its domain. It is discontinuous where the denominator $x^2 - 4x - 21 = (x+3)(x-7) = 0$ at $\boxed{x = -3, 7}$.

At $x = -3$,

$$\lim_{x\to -3} \frac{5x+15}{x^2-4x-21} = \lim_{x\to -3} \frac{5\cancel{(x+3)}}{\cancel{(x+3)}(x-7)} = \frac{5}{-10} = -\frac{1}{2}$$

so there is a $\boxed{\text{removable discontinuity}}$ at $x = -3$.

Approaching $x = 7$ from the right gives

$$\lim_{x\to 7^+} \frac{5x+15}{x^2-4x-21} = \lim_{x\to 7^+} \frac{5}{x-7} = \infty$$

because the denominator approaches 0 with positive values. Therefore there is an $\boxed{\text{infinite discontinuity}}$ at $x = 7$.

(b) The polynomial $P(x)$ is continuous for all x. By the Intermediate Value Theorem, because

$$\begin{aligned} P(0) &= 1 \quad \text{(a positive value)}, \\ P(1) &= 3 - 4 - 3 + 1 = -3 \quad \text{(a negative value)}, \\ P(2) &= 24 - 16 - 6 + 1 = 3 \quad \text{(a positive value)}, \end{aligned}$$

there must be a root in $(0, 1)$ and another root in $(1, 2)$.