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APPM 1350
Spring 2022

Final Exam

April 30
10:30 am - 1:00 pm

Instructions:

- This exam has 7 problems on pages numbered 1 through 11. Make sure you have all pages.
- Write your name and section number at the top of each page.
- Show all work and simplify your answers, except where the instructions tell you to leave your answer unsimplified.
- Name any theorem that you use and explain how it is used.
- Answers with no justification will receive no points unless the problem explicitly states otherwise.
- Notes, your text and other books, calculators, cell phones, and other electronic devices are not permitted, except as needed to upload your work.
- When you have completed the exam, go to the scanning section of the room and upload it to Gradescope. Verify that everything has been uploaded correctly and pages have been associated to the correct problem before you leave the room.
- Turn in your hardcopy exam before you leave the room.

Formulas

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + C \qquad \int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$$

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1. (24 pts) Evaluate the following limits.

(a) $\lim_{x\to\infty} \frac{\sinh(x)}{e^x}$ (b) $\lim_{r\to 1} \frac{r^8-1}{r^5-1}$ (c) $\lim_{x\to 2^+} \ln(x-2) - \ln(x^2-x-2)$

Solution:

(a) $\lim_{x\to\infty} \frac{\sinh(x)}{e^x} = \lim_{x\to\infty} \frac{\frac{1}{2}\left(e^x - e^{-x}\right)}{e^x} = \lim_{x\to\infty} \frac{1}{2}\left(1 - e^{-2x}\right) = \frac{1}{2}(1-0) = \boxed{\frac{1}{2}}$

(b) $\lim_{r\to 1} \frac{r^8-1}{r^5-1}$ is an indeterminate form of type $\frac{0}{0}$

$\lim_{r\to 1} \frac{r^8-1}{r^5-1} \overset{LHR}{=} \lim_{r\to 1} \frac{8r^7}{5r^4} = \lim_{r\to 1} \frac{8r^3}{5} = \boxed{\frac{8}{5}}$

(c) $\lim_{x\to 2^+} \ln(x-2) - \ln(x^2-x-2) = \lim_{r\to 2^+} \ln\left(\frac{x-2}{x^2-x-2}\right) = \lim_{r\to 2^+} \ln\left(\frac{x-2}{(x-2)(x+1)}\right) =$

$\lim_{r\to 2^+} \ln\left(\frac{1}{x+1}\right) = \boxed{\ln\left(\frac{1}{3}\right)} = \boxed{-\ln 3}$

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2. (32 pts) Find the requested information.

(a) Evaluate $\displaystyle\int \frac{3x^3+5x+7}{x^2}\,dx$

(b) Evaluate $\displaystyle\int_{\pi^2/16}^{\pi^2/4} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

(c) Let $g(x) = x\tan^{-1}(x^2)$. Find $g'(1)$.

(d) Let $f(t) = \dfrac{3-t}{4+t}$. Find $f''(1)$.

Solution:

(a) $$\int \frac{3x^3+5x+7}{x^2}\,dx = \int\left(3x+\frac{5}{x}+\frac{7}{x^2}\right)dx = \boxed{\frac{3}{2}x^2+5\ln|x|-\frac{7}{x}+C}$$

(b) Let $u=\sqrt{x}$, $du = dx/(2\sqrt{x})$. Then $x=\pi^2/16 \Rightarrow u=\pi/4$, and $x=\pi^2/4 \Rightarrow u=\pi/2$.

$$\begin{aligned}\int_{\pi^2/16}^{\pi^2/4} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx &= \int_{\pi/4}^{\pi/2} 2\sin u\,du = -2\cos u\Big|_{\pi/4}^{\pi/2}\\ &= -2[\cos(\pi/2)-\cos(\pi/4)] = -2\left(0-\sqrt{2}/2\right) = \boxed{\sqrt{2}}\end{aligned}$$

(c) $$g'(x) = \frac{2x^2}{1+x^4}+\tan^{-1}(x^2)$$

$$g'(1) = \boxed{1+\frac{\pi}{4}}$$

(d)

$$\begin{aligned}f'(t) &= \frac{-4-t-3+t}{(4+t)^2}\\ &= \frac{-7}{(4+t)^2}\\ f''(t) &= \frac{14}{(4+t)^3}\\ f''(1) &= \frac{14}{(5)^3}\\ &= \boxed{\frac{14}{125}}\end{aligned}$$

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3. (16 pts) A dose of a painkiller is administered to a patient. Suppose the rate at which the drug is eliminated from the body is proportional to the amount present, meaning we can model the amount of painkiller in a person's body by the function $P(t) = P_0 e^{kt}$ where k is a constant. Suppose the painkiller has a half-life of seven hours.

   (a) Find the decay constant k.

   (b) How long will it take for 90% of the drug to be eliminated?

Solution:

(a) Let $m(t) = m_0 e^{kt}$ represent the amount of painkiller in the body t hours after administration. Use the half-life to find k.
$m(t) = m_0 e^{kt} \Rightarrow m(7) = m_0 e^{7k} = \frac{1}{2} m_0 \Rightarrow e^{7k} = \frac{1}{2} \Rightarrow 7k = \ln(1/2) \Rightarrow k = \boxed{-(\ln 2)/7}$

(b) Now solve for t when 10% of the drug remains.
$m(t) = m_0 e^{kt} = \frac{1}{10} m_0 \Rightarrow e^{kt} = \dfrac{1}{10} \Rightarrow kt = \ln(1/10) \Rightarrow t = -\dfrac{\ln(10)}{k} = \boxed{\dfrac{7 \ln 10}{\ln 2} \text{ hrs}}$
≈ 23.3 hrs.

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4. (20 pts)

   (a) Find the average value of the function

$$f(x) = 8 - \cos\left(\frac{x}{4}\right)$$

   on the interval $[0, 4\pi]$.

   (b) Find all values of c that satisfy the Mean Value Theorem for integrals for the function and interval in part (a).

Solution:

(a) $f_{\text{avg}} = \dfrac{1}{4\pi}\displaystyle\int_0^{4\pi}\left(8 - \cos\left(\frac{x}{4}\right)\right)\,dx = \dfrac{1}{4\pi}\int_0^{4\pi} 8\,dx - \dfrac{1}{4\pi}\int_0^{4\pi}\cos\left(\frac{x}{4}\right)\,dx$

Let $u = \dfrac{x}{4}$, $du = \dfrac{dx}{4}$. Then $x = 4\pi \Rightarrow u = \pi$, and $x = 0 \Rightarrow u = 0$.

$$\begin{aligned} f_{\text{avg}} &= \frac{1}{4\pi}\int_0^{4\pi} 8\,dx - \frac{1}{4\pi}\int_0^{\pi} 4\cos u\,du \\ &= \frac{1}{4\pi}\left(8x\Big|_0^{4\pi}\right) + \frac{1}{\pi}\left(\sin u\Big|_0^{\pi}\right) \\ &= \frac{32\pi}{4\pi} + \frac{1}{\pi}(\sin\pi - \sin 0) \\ &= \boxed{8} \end{aligned}$$

(b)

$$\begin{aligned} f(c) &= 8 \\ 8 - \cos\left(\frac{c}{4}\right) &= 8 \\ \cos\left(\frac{c}{4}\right) &= 0 \\ \frac{c}{4} &= \pi n + \frac{\pi}{2},\ n \in \mathbb{Z} \\ c &= 4\pi n + 2\pi,\ n \in \mathbb{Z} \end{aligned}$$

The only solution in the interval $[0, 4\pi]$ is $\boxed{c = 2\pi}$.

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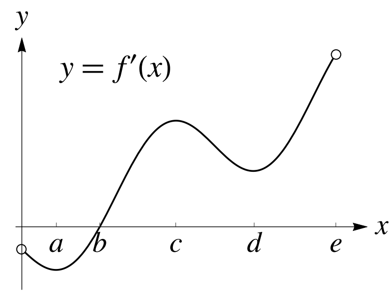
5. (24 pts)

The graph of the first derivative f' of a function f is shown at right. Answer the following questions about the function f which is defined on $[0, e]$. (List all answers that apply. Use interval notation where appropriate. No explanation is necessary.)



(a) On what intervals is f increasing?

(b) At what values of x does f have a local minimum value?

(c) On what intervals is f concave up?

(d) At what values of x does f have an inflection point?

Solution:

(a) $\boxed{(b, e)}$

(b) $x = \boxed{b}$

(c) $\boxed{(a, c), (d, e)}$

(d) $x = \boxed{a, c, d}$

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6. (14 pts) Two particles are moving along a line. Particle A's position is given by $A(t) = t^2 + 2 + 3\sin^2(t)$. Particle B's position is given by $B(t) = t - 3\cos^2(t)$. At what time are A and B closest to each other?

Solution:

Let $D(t)$ represent the distance between A and B at time t. Note that $A(0) = 2 > B(0) = -3$.

$$\begin{aligned} D(t) &= A(t) - B(t) \\ &= \left(t^2 + 2 + 3\sin^2(t)\right) - \left(t - 3\cos^2(t)\right) \\ &= t^2 - t + 2 + 3\left(\sin^2(t) + \cos^2(t)\right) \\ &= t^2 - t + 5 \\ \frac{dD}{dt} &= 2t - 1 \\ \frac{dD}{dt} &= 0 \implies t = 1/2 \end{aligned}$$

Since $d^2D/dt^2 = 2 > 0$, the function $D(t)$ is concave up and has a minimum value at $\boxed{t = 1/2}$.

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7. (20 pts)

(a) Find $\dfrac{dg}{dx}$ for the following function. You may leave your answer unsimplified.

$$g(x) = \int_{\sin^{-1}(x)}^{3} t\cos(t)\,dt.$$

(b) Given the function

$$y = x^{2x},$$

find the tangent line at the point $(1, 1)$. Write your answer in the form $y = mx + b$.

Solution:

(a) Use the FTC and the chain rule.

$$\frac{dg}{dx} = \frac{d}{dx}\int_{\sin^{-1}(x)}^{3} t\cos(t)\,dt = \frac{d}{dx}\int_{3}^{\sin^{-1}(x)} -t\cos(t)\,dt = -\sin^{-1}(x)\cos(\sin^{-1}(x))\frac{1}{\sqrt{1-x^2}} = -\sin^{-1}(x)\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} = \boxed{-\sin^{-1}(x)}$$

(b)

$$\begin{aligned}
y &= x^{2x}, \text{ note that } x > 0 \\
\ln y &= \ln\left(x^{2x}\right) \\
&= 2x\ln x \\
\frac{1}{y}\frac{dy}{dx} &= 2\ln x + 2 \\
\frac{dy}{dx} &= x^{2x}(2\ln x + 2) \\
&= 2x^{2x}(\ln x + 1) \\
\left.\frac{dy}{dx}\right|_{x=1} &= 2
\end{aligned}$$

$$\begin{aligned}
y - 1 &= 2(x-1) \\
y &= 2x - 2 + 1 \\
y &= 2x - 1
\end{aligned}$$

The tangent line to the graph of y at the point $(1, 1)$ is $\boxed{y = 2x - 1}$.

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Scratch work

Be sure to label your problems