Name ______________________ Section ________

APPM 1350
Spring 2022

Exam 3

April 6

Instructions:

- This exam has four problems on pages numbered 1 through 9. Make sure you have all pages.
- Write your name and section number at the top of each page.
- Show all work and simplify your answers, except where the instructions tell you to leave your answer unsimplified.
- Name any theorem that you use and explain how it is used.
- Answers with no justification will receive no points unless the problem explicitly states otherwise.
- Notes, your text and other books, calculators, cell phones, and other electronic devices are not permitted, except as needed to upload your work.
- When you have completed the exam, go to the scanning section of the room and upload it to Gradescope. Verify that everything has been uploaded correctly and pages have been associated to the correct problem before you leave the room.
- Turn in your hardcopy exam before you leave the room.

Formulas

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

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1. (40 pts)

   (a) Evaluate the integral if it exists.

   i. $\displaystyle\int \frac{3x+9}{\sqrt{x^2+6x}}\,dx$

   ii. $\displaystyle\int_{\frac{1}{2}}^{1} 2\pi\sin(\pi v)\cos(\pi v)\,dv$

   iii. $\displaystyle\int_{-3}^{3} (\,|2x-4|+3x\,)\ dx$ (Feel free to use a geometric formula.)

   (b) Find the average value of the function $f(x)=\cos(2x-\pi)$ on $\left[\frac{3\pi}{4},\,\pi\right]$.

   (c) If g is a continuous odd function on $[-5,5]$ and $\displaystyle\int_{-2}^{0} g(x)\,dx=-8$ and $\displaystyle\int_{0}^{5} g(x)\,dx=-6$, find the value of $\displaystyle\int_{-5}^{-2} g(x)\,dx$.

Solution:

(a) i. Let $u=x^2+6x \;\Rightarrow\; du=2(x+3)\,dx$.

$$\begin{aligned}\int \frac{3x+9}{\sqrt{x^2+6x}}\,dx &= \int \frac{3}{2\sqrt{u}}\,du\\ &= 3\sqrt{u}+C\\ &= 3\sqrt{x^2+6x}+C\end{aligned}$$

ii. Let $u=\sin(\pi v) \;\Rightarrow\; du=\pi\cos(\pi v)$.

$$\begin{aligned}\int_{\frac{1}{2}}^{1} 2\pi\sin(\pi v)\cos(\pi v)\,dv &= \int_{1}^{0} 2u\,du\\ &= u^2\Big|_{1}^{0}\\ &= -1\end{aligned}$$

iii.

$$\begin{aligned}\int_{-3}^{3} (\,|2x-4|+3x\,)\ dx &= \int_{-3}^{3} |2x-4|\,dx+\int_{-3}^{3} 3x\,dx\\ &= 2\int_{-3}^{3} |x-2|\,dx+0\\ &= 2\left(\frac{1}{2}(5)(5)+\frac{1}{2}(1)(1)\right)\\ &= 26\end{aligned}$$

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(b)

$$\frac{1}{\pi - \frac{3\pi}{4}} \int_{\frac{3\pi}{4}}^{\pi} \cos(2x - \pi)\, dx = \frac{4}{\pi} \cdot \frac{1}{2} \sin(2x - \pi) \Big|_{\frac{3\pi}{4}}^{\pi}$$
$$= \frac{2}{\pi} [0 - 1]$$
$$= -\frac{2}{\pi}$$

(c) Since g is odd $\int_{-5}^{5} g(x)\, dx = 0$.

$$\int_{-5}^{-2} g(x)\, dx + \int_{-2}^{0} g(x)\, dx + \int_{0}^{5} g(x) = 0$$
$$\int_{-5}^{-2} g(x)\, dx - 8 - 6 = 0$$
$$\int_{-5}^{-2} g(x)\, dx = 14$$

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2. (12 pts)

A fence is to be built to enclose a rectangular area of 250 square feet. The fence along three sides is to be made of material that costs 6 dollars per foot. The material for the fourth side will cost 10 dollars per foot. Find the dimensions of the enclosure that minimize the cost of fencing material.

Solution:

Let x be the width and y be the length of the enclosed area, and suppose that one of the sides of length y costs \$10. The two equations we have are:

$$A = xy = 250 \tag{1}$$

$$C = 6(2x) + 6y + 10y \tag{2}$$

Solving equation (1) for y gives $y = \dfrac{250}{x}$. Plugging this into the cost equation and optimizing:

$$\begin{aligned}
C &= 12x + 16\left(\frac{250}{x}\right) \\
C' = 0 &= 12 - \frac{16 \cdot 250}{x^2} \\
12 &= \frac{16 \cdot 250}{x^2} \\
x^2 &= \frac{16 \cdot 250}{12} \\
x &= \sqrt{\frac{16 \cdot 250}{12}} = 4 \cdot 5\sqrt{\frac{10}{12}} = 20\sqrt{\frac{5}{6}} \text{ or } 10\sqrt{\frac{10}{3}} \text{ or } \frac{10}{3}\sqrt{30} \text{ or } \frac{100}{\sqrt{30}} \\
y &= \frac{250}{\sqrt{\frac{16 \cdot 250}{12}}} = \frac{250\sqrt{12}}{\sqrt{16 \cdot 250}} = \frac{\sqrt{250 \cdot 12}}{\sqrt{16}} = \frac{5 \cdot 2\sqrt{10 \cdot 3}}{4} = \frac{5\sqrt{30}}{2}
\end{aligned}$$

Note that $C'' = \dfrac{2 \cdot 16 \cdot 250}{x^3} > 0$ for positive x, so the critical number we found will be a minimum. The enclosure will have dimensions of $\dfrac{5\sqrt{30}}{2}$ feet by $\dfrac{100}{\sqrt{30}}$ feet. One side of length $\dfrac{5\sqrt{30}}{2}$ feet will be made of material that costs 10 dollars per square foot.

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3. (24 pts)

   (a) Suppose an object moves with velocity $v(t) = 2t^2 - 12t + 16$ km/hr along a straight road.

      i. Determine the displacement of the object on the time interval [1,3].

      ii. Determine the distance traveled on the time interval [1,3].

   (b) Apply Newton's method to the equation $x^3 + x - 5 = 0$. Use an initial guess of $x_0 = 1$ and find x_1. (Find only x_1.)

Solution:

(a) i. The displacement is given by the integral of velocity over the time interval:

$$\int_1^3 2t^2 - 12t + 16\,dt = \frac{2}{3}t^3 - 6t^2 + 16t\Big|_1^3$$
$$= (2\cdot 9 - 6\cdot 9 + 16\cdot 3) - \left(\frac{2}{3} - 6 + 16\right)$$
$$= (-36 + 48) - \left(\frac{2}{3} + 10\right)$$
$$= 12 - 10 - \frac{2}{3} = \frac{4}{3}\,\text{km}$$

ii. The total distance travelled is given by the integral of the absolute value of velocity over the given interval. The velocity function factors as $v(t) = 2(t-2)(t-4)$, so on the interval $[1,2)$ the velocity is positive and on $(2,3]$ the velocity is negative.

$$\int_1^3 |2t^2 - 12t + 16|\,dt = 2\left[\int_1^2 t^2 - 6t + 8\,dt - \int_2^3 t^2 - 6t + 8\,dt\right]$$
$$= 2\left[\left(\frac{1}{3}t^3 - 3t^2 + 8t\right)\Big|_1^2 - \left(\frac{1}{3}t^3 - 3t^2 + 8t\right)\Big|_2^3\right]$$
$$= 2\left[\left(\frac{8}{3} - 3\cdot 4 + 8\cdot 2\right) - \left(\frac{1}{3} - 3 + 8\right)\right]$$
$$- 2\left[(9 - 3\cdot 9 + 8\cdot 3) - \left(\frac{8}{3} - 3\cdot 4 + 8\cdot 2\right)\right]$$
$$= 2\left[\left[\left(\frac{8}{3} + 4\right) - \left(\frac{1}{3} + 5\right)\right] - \left[(6) - \left(\frac{8}{3} + 4\right)\right]\right]$$
$$= 2\left[\left[\frac{7}{3} - 1\right] - \left[2 - \frac{8}{3}\right]\right]$$
$$= 2\,[5 - 3]$$
$$= 4\,\text{km}$$

(b) $f(x) = x^3 + x - 5$ and $f'(x) = 3x^2 + 1$.

$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &= 1 - \frac{1+1-5}{3+1} \\ &= 1 + 34 \\ &= \frac{7}{4} \end{aligned}$$

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4. (24 pts)

   (a) Evaluate the Riemann sum for $f(x) = x^2 - 3$ taking the sample points to be right endpoints, $a = -4$, $b = 2$ and $n = 6$.

   (b) Express the integral $\int_{-4}^{2} (x^2 - 3)\ dx$ as a limit of Riemann sums. You are not required to fully simplify this expression.

   (c) Evaluate the expression that you gave in (b). Show all steps to find the limit of the Riemann sums.

Solution:

(a) With $a = -4,\ b = 2,\ n = 6, \Delta x = \dfrac{2-(-4)}{6} = 1$. We make a table:

x	-4	-3	-2	-1	0	1	2
$f(x)$		6	1	-2	-3	-2	1

$R_6 = (1)(6 + 1 - 2 - 3 - 2 + 1) = 1.$

(b) and (c)

$$\Delta x = \frac{6}{n}$$

$$x_i = -4 + \frac{6i}{n}$$

$$f(x_i) = \left(-4 + \frac{6i}{n}\right)^2 - 3 = 13 - \frac{48i}{n} + \frac{36i^2}{n^2}$$

$$\begin{aligned}
\int_{-4}^{2} (x^2 - 3)\ dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \boxed{\lim_{n\to\infty} \frac{6}{n} \sum_{i=1}^{n} 13 - \frac{48i}{n} + \frac{36i^2}{n^2}} \\
&= \lim_{n\to\infty} \frac{6}{n} \left[\sum_{i=1}^{n} 13 - \frac{48}{n}\sum_{i=1}^{n} i + \frac{36}{n^2}\sum_{i=1}^{n} i^2\right] \\
&= \lim_{n\to\infty} \frac{6}{n} \left[13n - \frac{48}{n} \cdot \frac{n(n+1)}{2} + \frac{36}{n^2} \cdot \frac{n(n+1)(2n+1)}{6}\right] \\
&= \lim_{n\to\infty} 6 \left[13 - 24 \cdot \frac{n(n+1)}{n^2} + 6 \cdot \frac{n(n+1)(2n+1)}{n^3}\right] \\
&= 6\,[13 - 24 + 12] \\
&= \boxed{6}
\end{aligned}$$

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Scratch work

Be sure to label your problems.