Name ______________________________ Section __________

APPM 1350
Spring 2022

Exam 2

March 2

Instructions:

- Write your name and section number at the top of each page.
- Show all work and simplify your answers, except where the instructions tell you to leave your answer unsimplified.
- Name any theorem that you use and explain how it is used.
- Answers with no justification will receive no points unless the problem explicitly states otherwise.
- Notes, your text and other books, calculators, cell phones, and other electronic devices are not permitted, except as needed to upload your work.
- When you have completed the exam, go to the scanning section of the room and upload it to Gradescope. Verify that everything has been uploaded correctly and pages have been associated to the correct problem before you leave the room.
- Turn in your hardcopy exam before you leave the room.

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1. (30 pts)

   (a) Compute the derivatives for each of the following functions:

   i. $y = \dfrac{x^3 - 2x + 1}{2x + 2}$ ii. $y = (2 - \tan(3x - 1))^4$

   (b) Given the curve

   $$x\sqrt{y} = x - y + 5$$

   i. Find the derivative $\dfrac{dy}{dx}$ in terms of x and y. You do not need to simplify.

   ii. Find the tangent line to the above curve at the point $(1, 4)$.

Solution:

(a) i.

$$\begin{aligned} y' &= \frac{(2x+2)(3x^2-2) - (x^3-2x+1)(2)}{(2x+2)^2} \\ &= \frac{2(2x^3+3x^2-3)}{4(x+1)^2} \\ &= \frac{2x^3+3x^2-3}{2(x+1)^2} \end{aligned}$$

ii.

$$\begin{aligned} y' &= 4(2-\tan(3x-1))^3 \, \frac{d}{dx}(2-\tan(3x-1)) \\ &= 4(2-\tan(3x-1))^3 \, (-3)\sec^2(3x-1) \\ &= -12(2-\tan(3x-1))^3 \, \sec^2(3x-1) \end{aligned}$$

(b) i.

$$\begin{aligned} \frac{d}{dx} x\sqrt{y} &= \frac{d}{dx}(x-y+5) \\ \frac{x}{2\sqrt{y}}\frac{dy}{dx} + \sqrt{y} &= 1 - \frac{dy}{dx} \\ \left(\frac{x}{2\sqrt{y}} + 1\right)\frac{dy}{dx} &= 1 - \sqrt{y} \\ \frac{dy}{dx} &= \frac{1-\sqrt{y}}{\frac{x}{2\sqrt{y}} + 1} \end{aligned}$$

ii.

$$\begin{aligned} \left.\frac{dy}{dx}\right|_{x=1,y=4} &= \frac{1-2}{\frac{1}{4}+1} \\ &= -\frac{4}{5} \end{aligned}$$

The tangent line to the curve at the point (1,4) is the line through the point (1,4) with slope of $-\dfrac{4}{5}$, which is:

$$\begin{aligned} y - 4 &= -\frac{4}{5}(x-1) \\ y &= -\frac{4x}{5} + \frac{24}{5} \end{aligned}$$

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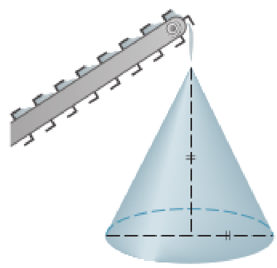
2. (15 pts) Gravel is being dumped from a conveyor belt at a rate of 10 cubic feet per minute. It forms a pile in the shape of a right circular cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 18 feet high? Recall that the volume of a right circular cone with height h and radius of the base r is given by $V = \frac{1}{3}\pi r^2 h$.



Solution: Note that $2r = h$, and equivalently, $r = \frac{h}{2}$.

$$\begin{aligned} V &= \frac{1}{3}\pi \frac{h^2}{4} h \\ &= \frac{\pi h^3}{12} \\ \frac{dV}{dt} &= \frac{\pi h^2}{4}\frac{dh}{dt} \end{aligned}$$

Given that $\frac{dV}{dt} = 10$, we can find $\frac{dh}{dt}$ when $h = 18$.

$$\begin{aligned} 10 &= \frac{\pi(18^2)}{4}\frac{dh}{dt} \\ \frac{dh}{dt} &= \frac{40}{324\pi} \\ &= \frac{10}{81\pi} \quad \text{feet per minute} \end{aligned}$$

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3. (20 pts)

   (a) Spherical ball bearings are manufactured to have a diameter of 1 ± 0.01 cm. Use differentials to estimate the maximum error in the volume. Recall that the volume of a sphere with radius r is given by $V = \frac{4}{3}\pi r^3$.

   (b) Let $k(x) = \frac{1}{\sqrt{x}}$. Use the limit definition of derivative to find $k'(4)$.

Solution:

(a) Let D = the diameter.

$$\begin{aligned} V &= \frac{\pi D^3}{6} \\ \frac{dV}{dD} &= \frac{\pi D^2}{2} \\ dV &= \frac{\pi D^2}{2} dD \end{aligned}$$

Then the maximum error in the volume is:

$$\begin{aligned} dV &= \frac{\pi 1^2}{2} \frac{1}{100} \\ &= \frac{\pi}{200} \quad \text{cubic centimeters} \end{aligned}$$

Alternative solution using the radius to calculate volume:

$$\begin{aligned} V &= \frac{4\pi r^3}{3} \\ \frac{dV}{dr} &= 4\pi r^2 \\ dV &= 4\pi r^2 dr \end{aligned}$$

Then the maximum error in the volume is:

$$\begin{aligned} dV &= 4\pi \left(\frac{1}{2}\right)^2 \frac{1}{200} \\ &= \frac{\pi}{200} \quad \text{cubic centimeters} \end{aligned}$$

(b)

$$
\begin{aligned}
k'(x) &= \lim_{h\to 0} \frac{k(x+h)-k(x)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}\right) \\
&= \lim_{h\to 0} \frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} \\
&= \lim_{h\to 0} \left(\frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}\right)\left(\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}\right) \\
&= \lim_{h\to 0} \frac{x-(x+h)}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} \\
&= \lim_{h\to 0} \frac{-h}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} \\
&= \lim_{h\to 0} \frac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} \\
&= \frac{-1}{2x\sqrt{x}} \\
&= -\frac{1}{2}x^{-3/2}
\end{aligned}
$$

$$k'(4) = -\frac{1}{16}$$

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4. (10 pts) Consider the function $f(x) = x^2 - x - 3$ on the interval $[-1, 4]$. For what values of c in $(-1, 4)$ is the conclusion of the Mean Value Theorem satisfied?

Solution:

Note that f is a polynomial and continuous and differentiable on $x \in \mathrm{R}$ and therefore continuous on $[-1, 4]$ and differentiable on $(-1, 4)$.

Therefore by the Mean Value Theorem, there is a number c in $(-1, 4)$ such that:

$$\begin{aligned} f'(c) &= \frac{f(4) - f(-1)}{4 - (-1)} \\ &= \frac{9 - (-1)}{5} \\ &= 2 \end{aligned}$$

We find c such that $f'(c) = 2$

$$\begin{aligned} f'(x) &= 2x - 1 \\ 2 &= 2x - 1 \\ x &= \frac{3}{2} \end{aligned}$$

Therefore the only value of c for which the conclusion of the MVT is satisfied is $c = \frac{3}{2}$.

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5. (25 pts) Suppose $f(x)$ is a twice differentiable function with

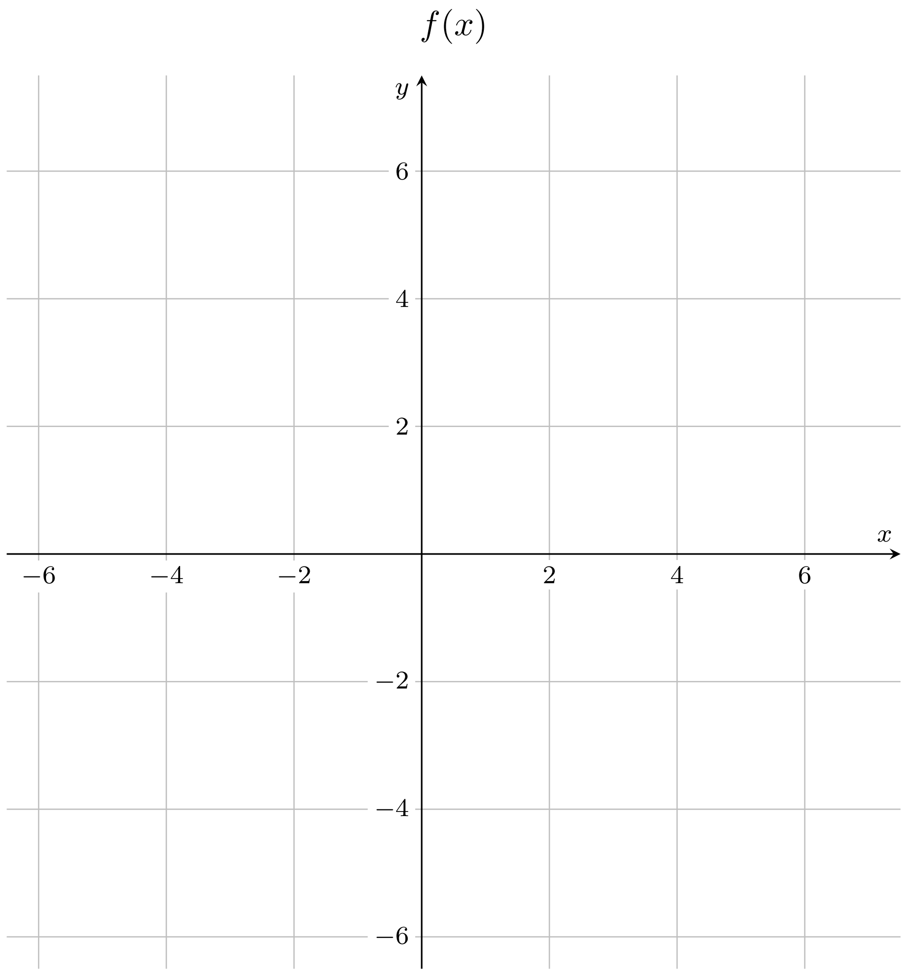
$$f'(x) = -2(x+1)^2(x-5).$$

(a) Find the intervals of increase and decrease of $f(x)$.

(b) Find and classify all local extrema.

(c) Find the intervals of concave up, concave down, and identify the position (x-value) of any inflection points.

(d) Graph a function $f(x)$ that meets the above criteria.



Solution:

(a) $f'(x) = 0$ when $x = -1$ and $x = 5$. Note that $f'(x)$ is a polynomial and continuous everywhere and therefore there are no other critical numbers of f. Since differentiability implies continuity, we know that f is continuous for $x \in \mathrm{R}$.

Testing points in the intervals, we find that $f'(-2) = 14 > 0$ and $f'(0) = 10 > 0$. Therefore, f is increasing on $(-\infty, -1)$ and $(-1, 5)$.

Note that it is correct to say that f is increasing on $(-\infty, 5)$; however, one must first examine each of the intervals $(-\infty, -1)$ and $(-1, 5)$.

Testing a point in the next interval, we find that $f'(6) = -98 < 0$. Therefore f is decreasing on $(5, \infty)$.

(b) Since $f'(x)$ changes from positive to negative at the critical number $x = 5$, then f has a local maximum at $x = 5$.

(c) $f''(x) = (-6)(x-3)(x+1)$ and $f''(x) = 0$ when $x = -1$ and $x = 3$.

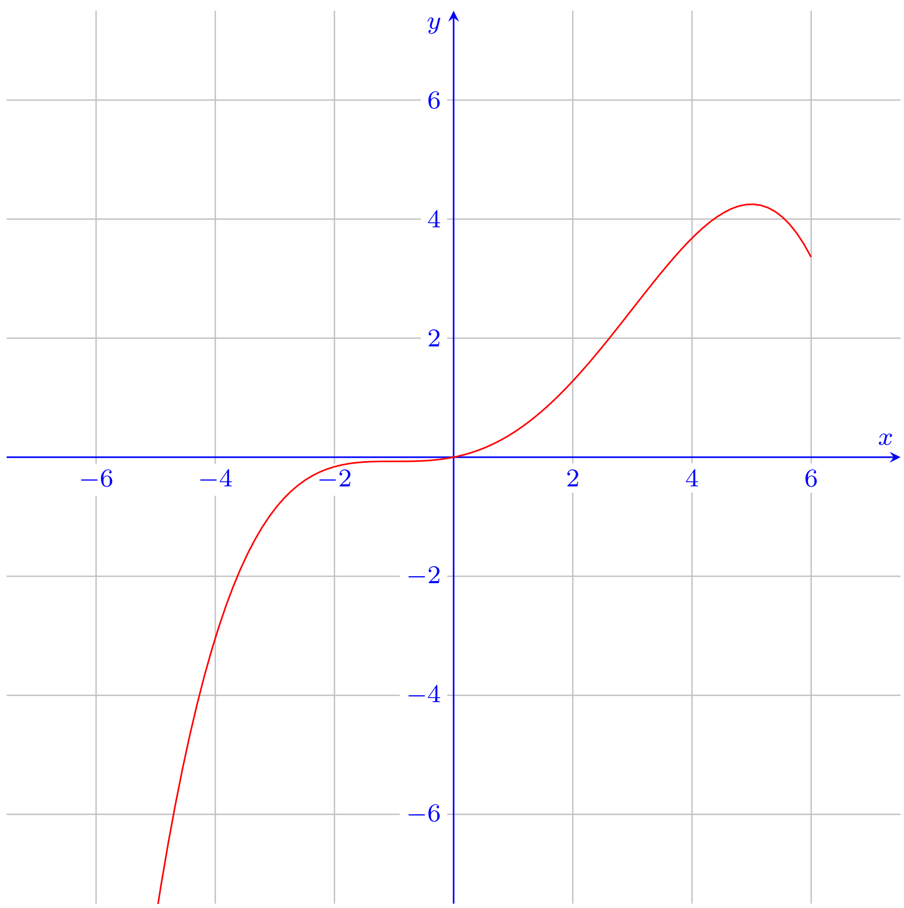
Testing points in the intervals, we find that:

$f''(-2) = -30 < 0$ and therefore f is concave down on $(-\infty, -1)$.
$f''(0) = 18 > 0$ and therefore f is concave up on $(-1, 3)$.
$f''(4) = -30 < 0$ and therefore f is concave down on $(3, \infty)$.

Since f is continuous at $x = -1$ and there is a change of concavity at $x = -1$ then there is an inflection point at $x = -1$.

Since f is continuous at $x = 3$ and there is a change of concavity at $x = 3$ then there is an inflection point at $x = 3$.

$f(x)$



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Scratch work

Be sure to label your problems