APPM 1350
Spring 2022

Exam 1

February 3

Instructions:

- Write your name and section number at the top of each page.
- Show all work and simplify your answers, except where the instructions tell you to leave your answer unsimplified.
- Name any theorem that you use and explain how it is used.
- Answers with no justification will receive no points unless the problem explicitly states otherwise.
- Notes, your text and other books, calculators, cell phones, and other electronic devices are not permitted, except as needed to upload your work.
- When you have completed the exam, email your proctor and request permission to begin scanning. Upload to Gradescope. Verify that everything has been uploaded correctly and pages have been associated to the correct problem before you leave the room.

Half / Double Angle Formulas

- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\cos(2\theta) = \begin{cases} \cos^2(\theta) - \sin^2(\theta) \\ 1 - 2\sin^2(\theta) \\ 1 + 2\cos^2(\theta) \end{cases}$
- $\tan(2\theta) = \dfrac{2\tan(\theta)}{1-\tan^2(\theta)}$
- $\sin\left(\dfrac{\theta}{2}\right) = \pm\sqrt{\dfrac{1}{2}(1-\cos(\theta))}$
- $\cos\left(\dfrac{\theta}{2}\right) = \pm\sqrt{\dfrac{1}{2}(1+\cos(\theta))}$
- $\tan\left(\dfrac{\theta}{2}\right) = \begin{cases} \pm\sqrt{\dfrac{1-\cos(\theta)}{1+\cos(\theta)}} \\ \dfrac{\sin(\theta)}{1+\cos(\theta)} \\ \dfrac{1-\cos(\theta)}{\sin(\theta)} \end{cases}$

Angle Sum / Difference Formulas

- $\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \sin(\beta)\cos(\alpha)$
- $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$
- $\tan(\alpha \pm \beta) = \dfrac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$

1. (20 pts)

   (a) Find all solutions to the following equation in $[0, 2\pi]$:

$$\sqrt{2}\sin(2x) + 1 = 0.$$

   (b) If $\tan(u) = \frac{3}{4}$ for u in $(\pi, \frac{3\pi}{2})$, find $\sin(u)$ and $\cos(u)$.

Solution:

(a)

$$\begin{aligned}
\sqrt{2}\sin(2x) + 1 &= 0 \\
\sin(2x) &= -\frac{1}{\sqrt{2}} \\
2x &= \begin{cases} \frac{5\pi}{4} + 2\pi k \\ \frac{7\pi}{4} + 2\pi k \end{cases} \\
x &= \begin{cases} \frac{5\pi}{8} + \pi k \\ \frac{7\pi}{8} + \pi k \end{cases}
\end{aligned}$$

On the interval of $[0, 2\pi]$ we get the following solutions:

$$x = \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}.$$

(b) For u in $(\pi, \frac{3\pi}{2})$, $\sin(u) < 0$ and $\sin(u) < 0$.

$\sin^2(u) + \cos^2(u) = 1$

$\frac{\sin(u)}{\cos(u)} = \frac{3}{4}$, equivalently, $\sin(u) = \frac{3}{4}\cos(u)$.

$$\begin{aligned}
\left(\frac{3}{4}\cos(u)\right)^2 + \cos^2(u) &= 1 \\
\frac{25}{16}\cos^2(u) &= 1 \\
\cos(u) &= \pm\frac{4}{5} \\
&= -\frac{4}{5} \qquad \text{since } u \in (\pi, \frac{3\pi}{2}) \\
\sin(u) &= \frac{3}{4}\cos(u) \\
&= \left(\frac{3}{4}\right)\left(\frac{-4}{5}\right) \\
&= -\frac{3}{5}
\end{aligned}$$

2. (20 pts)

(a) The intensity of a light follows the inverse square law

$$I(x) = \frac{k}{x^2}$$

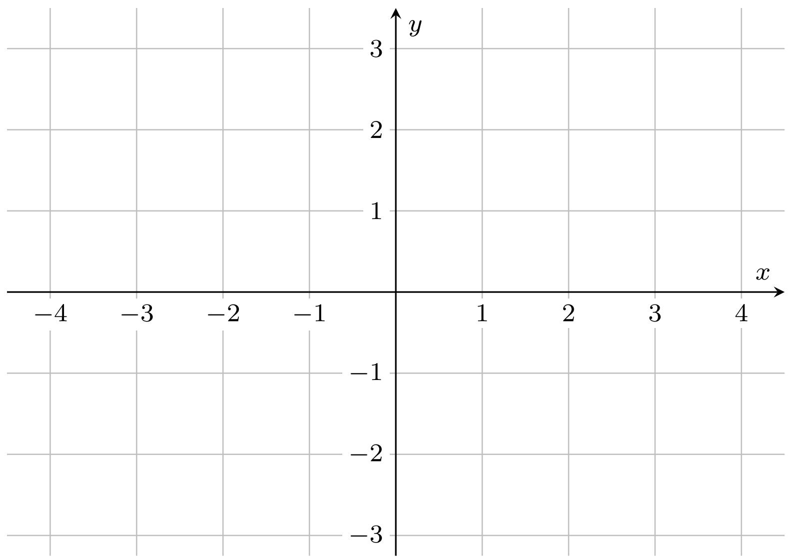
where x is the distance from the light in meters. If the intensity of the light at 3 meters is 20 lumens find the intensity of the light at 5 meters.

(b) The function $g(x)$ is obtained from the function $f(x) = \sqrt{x}$ by reflecting across the y-axis, shifting to the right by 2, then shifted down by 1.

i. Find a formula for the new function $g(x)$ obtained by the transformations of $f(x)$.

ii. Sketch a graph of f and sketch a graph of g on the same axes.

$f(x)$ and $g(x)$



Solution:

(a)

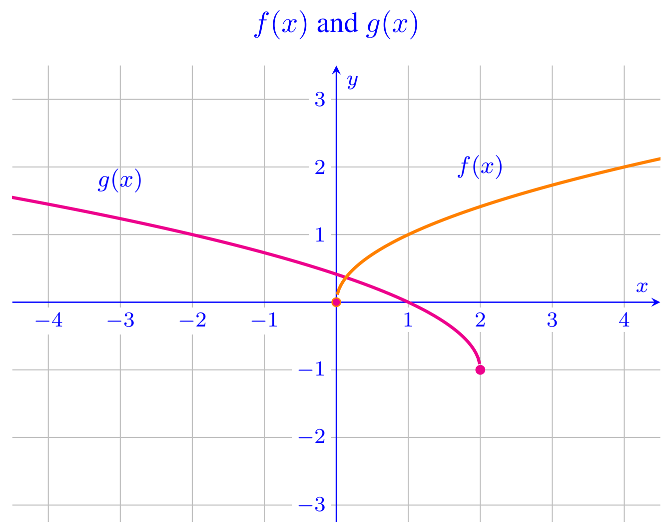
$$20 = \frac{k}{3^2}$$
$$180 = k$$

$$I(5) = \frac{180}{5^2} = \frac{36}{5}$$

(b)

i. $g(x) = \sqrt{-x+2} - 1$

Name ______________________________ **Section** __________



3. (26 pts) Evaluate the following limits or show that they do not exist.

(a) $\displaystyle\lim_{x\to -1} \frac{|x+1|}{2x+2}$

(b) $\displaystyle\lim_{x\to 0} \frac{3x\sec(x) - 4\sin(x)}{2\tan(x)}$

(c) $\displaystyle\lim_{x\to -3^-} \frac{x^2+2x+5}{2x+6}$

Solution:

(a) Note that the function is not defined at $x = -1$. We can write the function as a piecewise-defined function.

$$f(x) = \begin{cases} \dfrac{x+1}{2x+2} & x > -1 \\[2ex] \dfrac{-(x+1)}{2x+2} & x < -1 \end{cases}$$

$$= \begin{cases} \dfrac{x+1}{2(x+1)} & x > -1 \\[2ex] \dfrac{-(x+1)}{2(x+1)} & x < -1 \end{cases}$$

$$= \begin{cases} \dfrac{1}{2} & x > -1 \\[2ex] -\dfrac{1}{2} & x < -1 \end{cases}$$

In order for the limit to exist, the one-sided limits must be the same. However,

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} -\frac{1}{2} = -\frac{1}{2}$$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \frac{1}{2} = \frac{1}{2}$$

Therefore the limit as $x \to -1$ does not exist.

(b)

$$\begin{aligned}
\lim_{x\to 0} \frac{3x\sec(x) - 4\sin(x)}{2\tan(x)} &= \lim_{x\to 0} \frac{3x\frac{1}{\cos(x)} - 4\sin(x)}{2\frac{\sin(x)}{\cos(x)}} \\
&= \lim_{x\to 0} \frac{3x}{2\sin(x)} - 2\cos(x) \\
&= \frac{3}{2}\left(\lim_{x\to 0} \frac{x}{\sin(x)}\right) - \left(\lim_{x\to 0} 2\cos(x)\right) \\
&= \frac{3}{2} - 2 \\
&= -\frac{1}{2}
\end{aligned}$$

(c)

$$\begin{aligned}
\lim_{x\to -3^-} \frac{x^2+2x+5}{2x+6} &\approx \frac{8}{(\text{smaller and smaller}-)} \\
&= -\infty
\end{aligned}$$

In other words, if $f(x) = \dfrac{x^2+2x+5}{2x+6}$ and we check values closer and closer to the left of -3 for x:

$$\begin{aligned}
f(-3.1) &= \frac{(-3.1)^2 + 2(-3.1) + 5}{2(-3.1)+6} \approx \frac{9-6+5}{-0.2} \\
f(-3.01) &= \frac{(-3.01)^2 + 2(-3.01) + 5}{2(-3.01)+6} \approx \frac{9-6+5}{-0.02}
\end{aligned}$$

We see that the numerator will always be positive and at least 8, whereas the denominator will always be negative and approaching 0. So the limit tends to $-\infty$.

4. (34 pts)

(a) Consider the function $f(x) = \dfrac{2-x}{\sqrt{4x+1}-3}$

i. Find the domain of this function.

ii. Show that there is a removable discontinuity at $x = 2$. Use appropriate limit notation.

(b) Show that the following equation has at least one real solution.

$$\sin(x) = 7\cos(x)$$

(c) Consider $w(x) = \dfrac{1+4x}{x}$ and $v(x) = \dfrac{1}{x-3}$. Find $w \circ v$, simplify as much as possible and give its domain.

Solution:

(a) i. $4x+1 \geq 0$ implies $x \geq -\dfrac{1}{4}$. Excluded from the domain is the point at which $\sqrt{4x+1}-3 = 0$, which excludes $x = 2$. Therefore the domain of the function is $\Big[-\dfrac{1}{4}, 2\Big) \cup \Big(2, \infty\Big)$ or equivalently, $\{x \in \mathbb{R} : x \geq -\dfrac{1}{4}, x \neq 2\}$.

ii.

$$\begin{aligned}
\lim_{x\to 2} \frac{2-x}{\sqrt{4x+1}-3} &= \lim_{x\to 2} \Big(\frac{2-x}{\sqrt{4x+1}-3}\Big)\Big(\frac{\sqrt{4x+1}+3}{\sqrt{4x+1}+3}\Big) \\
&= \lim_{x\to 2} \frac{(2-x)(\sqrt{4x+1}+3)}{4x+1-9} \\
&= \lim_{x\to 2} \frac{(2-x)(\sqrt{4x+1}+3)}{-4(2-x)} \\
&= \lim_{x\to 2} \frac{(\sqrt{4x+1}+3)}{-4} \\
&= \frac{\sqrt{9}+3}{-4} \\
&= -\frac{3}{2}
\end{aligned}$$

Since the limit as $x \to 2$ = $-\dfrac{3}{2}$ and since the function f is not defined at the point $(2, -\dfrac{3}{2})$, then there is a removable discontinuity at $x = 2$.

(b) Let $f(x) = \sin(x) - 7\cos(x)$ Note that the function is continuous over the real line since both $g(x) = \sin(x)$ and $h(x) = -7\cos(x)$ are continuous over the real line and the sum of two functions that are continuous over an interval is continuous over the interval. $f(0) = -7$ and $f(\pi/2) = 1$. The function f is continuous over $[0, \pi/2]$ since it is continuous over the real line. Note that $f(0) < 0 < f(\pi/2)$. Therefore, by the IVT there exists a c in $(0, \pi/2)$ such that $f(c) = 0$ and at this c we have $\sin(c) = 7\cos(c)$.

(c)

$$\begin{aligned} w \circ v &= \frac{1 + \frac{4}{x-3}}{\frac{1}{x-3}} \\ &= \frac{\frac{x-3+4}{x-3}}{\frac{1}{x-3}} \\ &= x + 1 \quad \text{for } x \neq 3 \end{aligned}$$

The domain of $w \circ v$ is $\{x \in \mathbb{R} : x \neq 3\}$ or equivalently $(-\infty, 3) \cup (3, \infty)$.

The End