1. (34pts) The following parts of this problems are not related.

(a)(17pts) Use the Quotient Rule to find the derivative of $f(x) = \dfrac{5x-1}{\ln(x)}$. Simplify your answer.

(b)(17pts)(i)(5pts) Consider the limit $\lim\limits_{x\to\infty} (5x)^{1/x}$. What type of indeterminate form is this limit?

(ii)(12pts) Evaluate the limit $\lim\limits_{x\to\infty} (5x)^{1/x}$. Show all work.

Solution: (a)(17pts) Note that, by the Quotient Rule, we have

$$\left[\frac{5x-1}{\ln(x)}\right]' = \underbrace{\frac{5\cdot\ln(x) - (5x-1)\cdot\frac{1}{x}}{[\ln(x)]^2}}_{\text{Quotient Rule}}\cdot\frac{x}{x} = \boxed{\frac{5x\ln(x)-5x+1}{x\ln^2(x)}.}$$

(b)(i)(5pts) Applying the limit yields the indeterminate form $\boxed{\text{“}\infty^0\text{”}.}$

(b)(ii)(12pts) Note that using using $a = e^{\ln(a)}$ and continuity, we have

$$\lim_{x\to\infty}(5x)^{1/x} = \lim_{x\to\infty} e^{\ln\left[(5x)^{1/x}\right]} = \lim_{x\to\infty} e^{\frac{\ln(5x)}{x}} \overset{c}{=} e^{\lim\limits_{x\to\infty}\frac{\ln(5x)}{x}} \overset{L'H}{=} e^{\lim\limits_{x\to\infty}\frac{\frac{5}{5x}}{1}} = e^{\lim\limits_{x\to\infty}\frac{1}{x}} = e^0 = 1 \Rightarrow \boxed{\lim_{x\to\infty}(5x)^{1/x} = 1.}$$

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2. (35pts) The following parts of this problems are not related. **Start this problem on a new page.**

(a)(16pts) Find the linearization of $f(x) = \arctan(x)$ at the point $a = 1$. Show all work.

(b)(16pts) Find the antiderivative: $\displaystyle\int \frac{3}{\sqrt{1-9x^2}}\,dx$. Show all work.

(c)(3pts) For which choice of the number c will the function $f(x) = \begin{cases} cx - \dfrac{1}{5}, & \text{if } x < 2, \\ \dfrac{x^3-7x^2+10x}{x^2+x-6}, & \text{if } x \geq 2, \end{cases}$ be continuous?

(**No justification necessary** - *Choose only* <u>*one*</u> *answer, copy down the entire answer.*)

(A) $c=2$ (B) $c=-\dfrac{1}{2}$ (C) $c=-\dfrac{4}{3}$ (D) $c=5$ (E) None of these

Solution: (a)(16pts) We need to find $L(x) = f(1) + f'(1)(x-1)$ where $f(x) = \arctan(x)$, thus

$$f(x) = \arctan(x) \Rightarrow f'(x) = \frac{1}{1+x^2} \Rightarrow f'(1) = \left.\frac{1}{1+x^2}\right|_{x=1} = \frac{1}{1+1} = \frac{1}{2}$$

and so the linearization is

$$L(x) = f(1) + f'(1)\,(x-1) = \arctan(1) + \frac{1}{2}\cdot(x-1) = \frac{\pi}{4} + \frac{x}{2} - \frac{1}{2}. \Rightarrow \boxed{L(x) = \frac{\pi}{4} - \frac{1}{2} + \frac{x}{2}.}$$

(b)(16pts) Note that if we let $u = 3x \Rightarrow du = 3dx$ and so

$$\int \frac{3}{\sqrt{1-9x^2}}\,dx = \int \frac{1}{\sqrt{1-(3x)^2}}\,3dx = \int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}(u) + C = \boxed{\sin^{-1}(3x) + C.}$$

(c)(3pts) $\boxed{\text{Choice (B).}}$ <u>Discussion:</u> We need the two-sided limit at $x = 2$ to exist and note that

$$f(2) = \lim_{x\to 2^+} f(x) = \underbrace{\lim_{x\to 2^+}\frac{x^3-7x^2+10x}{x^2+x-6}}_{0/0} \overset{L'H}{=} \lim_{x\to 2^+}\frac{3x^2-14x+10}{2x+1} = \frac{12-28+10}{4+1} = -\frac{6}{5}$$

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and so, for continuity at $x = 2$, we need

$$\lim_{x\to 2^-} f(x) = -\frac{6}{5} \Leftrightarrow 2c - \frac{1}{5} = -\frac{6}{5} \Rightarrow 2c = -1 \Rightarrow c = -\frac{1}{2} \Rightarrow \text{choice (B).}$$

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3. (34pts) The following parts of this problems are not related. **Start this problem on a new page.**

(a)(17pts) Use *logarithmic differentiation* to find $\frac{dy}{dx}$ if $y = x^{\sin(x)}$. Your final answer should be in terms of the x-variable.

(b)(17pts) Given that the graph of $f(x) = 1 + 2x + \sinh^3(x)$ is one-to-one, find $(f^{-1})'(1)$. Justify.

Solution: (a)(17pts) Note that

$$\begin{aligned} y = x^{\sin(x)} \quad &\Rightarrow \quad \ln(y) = \ln\left(x^{\sin(x)}\right) = \sin(x)\ln(x) \\ &\overset{d/dx}{\Rightarrow} \quad \frac{y'}{y} = \cos(x)\ln(x) + \sin(x)\cdot\frac{1}{x} \\ &\overset{d/dx}{\Rightarrow} \quad y' = y\left[\cos(x)\ln(x) + \frac{\sin(x)}{x}\right] \Rightarrow \boxed{\frac{dy}{dx} = x^{\sin(x)}\left[\cos(x)\ln(x) + \frac{\sin(x)}{x}\right].} \end{aligned}$$

ALTERNATE METHOD Recall that $a = e^{\ln(a)}$ thus

$$y = x^{\sin(x)} = e^{\ln\left(x^{\sin(x)}\right)} = e^{\sin(x)\ln(x)} \overset{d/dx}{\Rightarrow} \frac{dy}{dx} = e^{\sin(x)\ln(x)}[\sin(x)\ln(x)]' \Rightarrow \boxed{\frac{dy}{dx} = x^{\sin(x)}\left[\cos(x)\ln(x) + \frac{\sin(x)}{x}\right].}$$

(b)(17pts) By a theorem in class we know that

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} \text{ where here } f'(x) = [1 + 2x + \sinh^3(x)]' = 2 + 3\sinh^2(x)\cosh(x)$$

and, by observation, $f(0) = 1$ which implies $0 = f^{-1}(1)$ thus

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{2 + 3\sinh^2(0)\cosh(0)} = \frac{1}{2+0} \Rightarrow \boxed{(f^{-1})'(1) = \frac{1}{2}.}$$

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4. (35pts) The following parts of this problems are not related. **Start this problem on a new page.**

(a)(16pts) Find the derivative of $g(x) = \int_e^{e^{2x}} \ln(t)\,dt$. Show all work and simplify your answer.

(b)(16pts) Evaluate the definite integral: $\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx$. Show all work.

(c)(3pts) Which graph below best matches the graph of the function $f(x) = \frac{e}{e^x} + 1$? **Choose only <u>one</u> answer.** *No justification necessary, clearly indicate your answer otherwise points will be deducted.*

Solution: (a)(16pts) By the Fundamental Theorem of Calculus (Part I), we have

$$g'(x) = \frac{d}{dx}\left[\int_e^{e^{2x}} \ln(t)\,dt\right] = \ln(e^{2x})\cdot[e^{2x}]' = 2x\cdot 2e^{2x} = \boxed{4xe^{2x}.}$$

(b)(16pts) If we let $u = x^2 \Rightarrow du = 2xdx \Rightarrow \frac{du}{2} = xdx$ and $x = 0 \Rightarrow u = 0$ and $x = \sqrt{\pi} \Rightarrow u = \pi$ thus we have

$$\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx = \int_0^{\sqrt{\pi}} \cos(x^2)\,xdx = \frac{1}{2}\int_0^{\pi} \cos(u)\,du = \left.\frac{\sin(u)}{2}\right|_0^{\pi} = \boxed{0.}$$

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(c)(3pts) Graph (B). Discussion: Note that $f(x) = \frac{e}{e^x} + 1 = e^{1-x} + 1 = e^{-(x-1)} + 1$ so we need to reflect the graph of e^x about the y-axis (so we can eliminate graphs (A) and (D) since they are increasing) and then shift to the right one unit and then translate the graph up one unit thus the only possibility is graph (B) (we can eliminate graph (C) since it goes below the x-axis).

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5. (12pts) Answer either **ALWAYS TRUE** or **FALSE**. You do NOT need to justify your answer. (*Don't just write down "A.T." or "F", completely write out the words "ALWAYS TRUE" or "FALSE" depending on your answer.*)

(a)(3pts) The sum $4 + 6 + 8 + 10 + 12$ can be written as $\sum_{i=11}^{15} [2i - 18]$.

(b)(3pts) We can show that $\lim_{h \to 0} \frac{5^{2+h} - 25}{h} = 25 \ln(5)$.

(c)(3pts) If, for an object moving in a straight line, we have $a(t) = 10 \frac{\text{m}}{\text{s}^2}$, $v(0) = 5 \frac{\text{m}}{\text{s}}$ and $s(0) = 0$, then $s(2) = 40\text{m}$.

(d)(3pts) Note that $\int_{-1}^{1} |2x| dx = |x^2| \Big|_{-1}^{1} = 1^2 - (-1)^2 = 0$.

Solution: 3pts each: (a) ALWAYS TRUE (b) ALWAYS TRUE (c) FALSE (d) FALSE

Discussion:

(a) By a direct application of the sigma notation we have

$$\sum_{i=11}^{15} [2i - 18] = [22 - 18] + [24 - 18] + [26 - 18] + [28 - 18] + [30 - 18] = 4 + 6 + 8 + 10 + 12 \Rightarrow \text{A.T.}$$

(b) By definition of the derivative, if $f(x) = 5^x$ then

$$\lim_{h \to 0} \frac{5^{2+h} - 25}{h} = \lim_{h \to 0} \frac{5^{2+h} - 5^2}{h} = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = f'(2) = 5^x \ln(5) \Big|_{x=2} = 5^2 \ln(5) = 25 \ln(5) \Rightarrow \text{A.T.}$$

(c) Note that

$$a(t) = 10 \Rightarrow v(t) = \int a(t)\, dt = 10t + v_0 \Rightarrow s(t) = \int v(t)\, dt = 5t^2 + v_0 t + s_0$$

and $v_0 = v(0) = 5$ and $s_0 = s(0) = 0$ so $s(t) = 5t^2 + 5t + 0 \Rightarrow s(2) = 5(4) + 10 = 30 \Rightarrow$ F.

(d) Here, since the integrand is even, we have

$$\int_{-1}^{1} |2x| dx = 2 \int_{0}^{1} 2x dx = 2x^2 \Big|_{0}^{1} = 2 \Rightarrow \text{F.}$$

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