1. (28pts) Start this problem on a **new** page. The following parts of this problem are not related.

(a)(12pts) Find the equation of the *tangent line* to $f(x) = x^3 + 2\sqrt{x}$ at $x = 1$. Simplify your answer.

(b)(12pts) Given $F(x) = \dfrac{2x^3}{4x+1}$, find the derivative using the *Quotient Rule*. Simplify the numerator of your answer.

(c)(4pts) *(Multiple Choice)* Use *differentials* to identify the maximum error of the volume of a cube whose sides are 10 ft in length, if the maximum error of the length is 0.5 ft. (**No justification necessary** - *Choose only one answer from below, copy down the entire answer.*)

(A) $125\,\text{ft}^3$ (B) $300\,\text{ft}^3$ (C) $50\,\text{ft}^3$ (D) $150\,\text{ft}^3$ (E) None of these

Solution: (a)(12pts) We need to find $y = f(1) + f'(1)(x-1)$. Note that $f'(x) = 3x^2 + x^{-1/2} \Rightarrow f'(1) = 4$, thus we have

$$y = f(1) + f'(1)(x-1) = 3 + 4(x-1) = 4x - 1 \;\Rightarrow\; \boxed{y = 4x - 1.}$$

(b)(12pts) If we let $f(x) = 2x^3$ and $g(x) = 4x+1$, then $f'(x) = 6x^2$ and $g'(x) = 4$ thus, by quotient rule, we have

$$F'(x) = \left[\frac{2x^3}{4x+1}\right]' = \frac{6x^2\cdot(4x+1) - 2x^3\cdot 4}{(4x+1)^2} = \frac{24x^3 + 6x^2 - 8x^3}{(4x+1)^2} \boxed{= \frac{16x^3 + 6x^2}{(4x+1)^2} = \frac{2x^2(8x+3)}{(4x+1)^2}.}$$

(c)(4pts) $\boxed{\text{Choice (D).}}$ Discussion: Note that $V(x) = x^3$ and $x = 10$ while $dx = 0.5$, thus,

$$\Delta V \approx dV = 3x^2 dx = 3(10^2)(1/2) = 150\,\text{ft}^3 \;\Rightarrow\; \text{choice (D).}$$

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2. (24pts) Start this problem on a **new** page. The following parts of this problem are not related.

(a)(12pts) For what values of a and b will the function $f(x) = \begin{cases} ax^2 - 3, & \text{if } x \le -1 \\ a + bx, & \text{if } x > -1 \end{cases}$ be *differentiable* at $x = -1$? Justify your answer.

(b)(12pts) Find the derivative of $f(x) = 2\sin\big(3x + \tan(x)\big)$ at $x = 0$. Show all work.

Solution: (a)(12pts) In order to be differentiable, we first need to have a continuous function, that is, we need $\lim_{x \to -1} f(x) = f(-1)$, but note that

$$\lim_{x \to -1^-} f(x) = ax^2 - 3 = a(-1)^2 - 3 = a - 3 = f(-1) \text{ and } \lim_{x \to -1^+} f(x) = a + bx = a + b(-1) = a - b,$$

so $\boxed{\lim_{x \to -1} f(x) = f(-1) \Rightarrow a - 3 = a - b, \text{ so } b = 3.}$ Additionally, we need to find the value of a that makes the function differentiable. Note that the function $y = ax^2 - 3$ and $y = a + bx$ are each differentiable at $x = -1$ and so the one-sided derivatives of the piecewise defined function at $x = -1$ are equal to the (two-sided) derivatives of the functions $y = ax^2 - 3$ and $y = a + bx$ at $x = -1$, that is, for differentiability at $x = -1$, we need

$$\lim_{h \to 0^-} \frac{f(-1+h) - f(-1)}{h} = \lim_{h \to 0^+} \frac{f(-1+h) - f(-1)}{h} \Rightarrow 2ax\Big|_{x=-1} = b\Big|_{x=-1} \Rightarrow \boxed{-2a = b.}$$

Thus, $-2a = 3 \Rightarrow a = -3/2$ and so the function is $\boxed{\text{differentiable when } a = -3/2 \text{ and } b = 3.}$

(b)(12pts) Taking this derivative using the chain rule, we get

$$f'(x) = 2\cos\big(3x + \tan(x)\big) \cdot \big(3 + \sec^2(x)\big) \Rightarrow f'(0) = 2\cos(0+0) \cdot (3 + 1^2) = \boxed{8.}$$

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3. (24pts) Start this problem on a **new** page. The following parts of this problem are not related.

(a)(10pts) Find the *absolute* minimum and maximum values of $f(x) = \sqrt{x}(x-6) = x^{3/2} - 6x^{1/2}$ on the interval $[0, 4]$. Give your answer in the form (x, y). Show all work, justify your answers and clearly label your answers.

(b)(10pts) Find dy/dx by *implicit differentiation* given that $x^3 - 2y = xy^2$. Simplify your answer.

(c)(4pts) *(Multiple Choice)* If $u(x)$ is a differentiable function of x such that $u(2) = -5 = u'(2)$ and if $f(u) = u^2$ then $\left.\frac{df}{dx}\right|_{x=2}$ is equal to which choice below? (**No justification necessary** - *Choose only* <u>*one*</u> *answer, copy down the entire answer.*)

(A) 25 (B) 50 (C) 125 (D) -125 (E) None of these

Solution: (a)(10pts) First we find the critical points, note that

$$f'(x) = [\sqrt{x}(x-6)]' = [x^{3/2} - 6x^{1/2}]' = \frac{3}{2}x^{1/2} - \frac{6}{2}x^{-1/2} = \frac{3x - 6}{2x^{1/2}}$$

so that $f'(x) = 0 \Rightarrow 3x - 6 = 0 \Rightarrow x = 2$ and $f'(x)$ is undefined when $x = 0$. Thus the critical points are $x = 0, 2$. So, on the closed interval $[0, 4]$, we have

$$f(0) = 0, \quad f(2) = \sqrt{2}(2-6) = -4\sqrt{2}, \text{ and } f(4) = \sqrt{4}(4-6) = 2 \cdot -2 = -4$$

thus there is an absolute maximum value at $(0,0)$ and an absolute minimum value at $(2,-4\sqrt{2})$.
(Note that $2>1>0 \Rightarrow \sqrt{2}>\sqrt{1}>0 \Rightarrow -4\sqrt{2}<-4<0$.)

(b)(10pts) Assuming $y=f(x)$ and differentiating both sides with respect to x yields

$$x^3-2y=xy^2 \overset{d/dx}{\Rightarrow} 3x^2-2y'=1\cdot y^2+x\cdot 2yy' \;\Rightarrow\; (2xy+2)y'=3x^2-y^2 \;\Rightarrow\; \boxed{y'=\frac{3x^2-y^2}{2(xy+1)}}.$$

(c)(4pts) Choice (B). <u>Discussion:</u> Note that $y=f(u)=f(u(x))$ so, by Chain Rule, we have

$$\left.\frac{df}{dx}\right|_{x=2}=\frac{d}{dx}[f(u(x))]\Big|_{x=2}=f'(u(x))\cdot u'(x)\Big|_{x=2}=f'(u(2))\cdot u'(2)=f'(-5)\cdot -5=2(-5)\cdot -5=50$$

where we have used the fact that $f'(u)=2u$. So the answer is choice (B).

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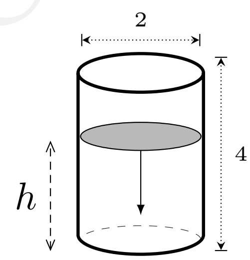
4. (24pts) Start this problem on a **new** page. The following parts of this problem are not related.

(12pts) (a)(i)(6pts) Explain why the function $f(x)=x^3-3x+2$ satisfies the hypothesis of the Mean Value Theorem on the interval $[-2,2]$. (a)(ii)(6pts) Find all numbers $c\in(-2,2)$ that satisfy the conclusion of the Mean Value Theorem. **Explanations should be in complete sentences**, show all work.

(b)(12pts) A water heater that has the shape of a cylindrical tank with a diameter of 2 m and a height of 4 m is being drained (see the diagram on the right). How fast is water draining out of the tank (in m^3/min) if the water level is dropping at 50 cm/min? **Be sure to answer the question being asked and give your final answer in the form of a complete sentence.**



Solution:

(a)(12pts) Note that $f(x)=x^3-3x+2$ is **continuous** on $[-2,2]$ and **differentiable** on $(-2,2)$ since **polynomials** are continuous and differentiable on $\mathbb{R}$ (by the work done in class) so the Mean Value Theorem applies. Note that

$$f'(c)=\frac{f(2)-f(-2)}{2-(-2)} \;\Rightarrow\; 3c^2-3=\frac{4-0}{4} \;\Rightarrow\; 3c^2=4 \;\Rightarrow\; c^2=\frac{4}{3} \;\Rightarrow\; \boxed{c=\pm\sqrt{\frac{4}{3}}=\pm\frac{2}{\sqrt{3}}=\pm\frac{2\sqrt{3}}{3}}.$$

(Note that $0<1<3\Rightarrow 0<\sqrt{1}<\sqrt{3}\Rightarrow 0<\frac{1}{\sqrt{3}}<1$ which implies $0<\frac{2}{\sqrt{3}}<2$ and $0>-\frac{2}{\sqrt{3}}>-2$.)

(b)(12pts) We are given that the diameter is <u>fixed</u> at $2r=2\,\text{m}\Rightarrow r=1$ m and since the height of the water is decreasing we have $\frac{dh}{dt}=-50\,\text{cm/min}=-0.5\,\text{m/min}$ and we wish to find the rate of change of the volume dV/dt where $V=\pi r^2h=\pi\cdot 1^2\cdot h=\pi h$ thus

$$V=\pi h \;\Rightarrow\; \frac{dV}{dt}=\pi\frac{dh}{dt}=-0.5\pi\,\text{m}^3/\text{min}=-\frac{\pi}{2}\,\text{m}^3/\text{min}$$

thus the water is draining **out** of the tank at $0.5\pi\,\text{m}^3/\text{min}$.
