1. (24pts) The following parts of this problem are not related.

(a)(12pts) Given the functions \( f(x) = \frac{1}{x^2 - 3} \) and \( g(x) = \sqrt{x + 1} \), find the composition \((f \circ g)(x)\) and state the domain using interval notation.

(b)(12pts) Evaluate the limit: \( \lim_{x \to 1} \frac{x^2 + 4x - 5}{x^2 - x} \). Show all work.

Solution: (a)(12pts) Note that the composition is
\[
(f \circ g)(x) = f(g(x)) = \frac{1}{(\sqrt{x + 1})^2 - 3} = \frac{1}{x + 1 - 3} = \frac{1}{x - 2}
\]
and for \( f(g(x)) \) to be well defined we need \( x \neq 2 \) and the domain of \( g(x) \) is \([-1, \infty)\), so the domain of \( f(g(x)) \) in interval notation is \([-1, 2) \cup (2, \infty)\).

(b)(12pts) Applying the limit directly gives a “0/0” type indeterminate form and factoring yields
\[
\lim_{x \to 1} \frac{x^2 + 4x - 5}{x^2 - x} = \lim_{x \to 1} \frac{(x-1)(x+5)}{x(x-1)} = \lim_{x \to 1} \frac{x+5}{x} = \frac{1+5}{1} = 6.
\]

2. (28pts) Start this problem on a new page. The following parts of this problem are not related.

(a)(12pts) Evaluate the limit: \( \lim_{x \to 4} \frac{\sqrt{6x+1} - 5}{x - 4} \). Show all work.

(b)(12pts) Suppose \( f(x) = \begin{cases} 
  x^2 + x, & \text{if } x \neq 0 \\
  \cos(x), & \text{if } x = 0
\end{cases} \). (i)(6pts) Find the limit \( \lim_{x \to 0} f(x) \). (ii)(6pts) Is \( f(x) \) continuous for all real \( x \)? If not, classify the discontinuities of \( f(x) \). Be sure to show that all three conditions of continuity have been satisfied and justify your answer.

(c)(4pts) The function \( f(x) = \frac{3x + 1}{\sqrt{8x^3} + 5} \) has a horizontal asymptote at which choice below? (No justification necessary - Choose only one answer, copy down the entire answer.)

(A) \( y = 0 \)  (B) \( y = \frac{3}{2} \)  (C) \( y = 0 \) and \( y = 3/2 \)  (D) \( y = -3/2 \) and \( y = 3/2 \)  (E) None of these

Solution:

(a)(12pts) Multiplying by the conjugate yields:
\[
\lim_{x \to 4} \frac{\sqrt{6x+1} - 5}{x - 4} = \lim_{x \to 4} \frac{(\sqrt{6x+1} - 5)(\sqrt{6x+1} + 5)}{(x - 4)(\sqrt{6x+1} + 5)} = \lim_{x \to 4} \frac{6(x-4)}{(x-4)(\sqrt{6x+1} + 5)} = \frac{6}{10} = \frac{3}{5}.
\]
(b)(i) (6pts) Note that
\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 + x = 0 \quad \text{and} \quad \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 + x = 0 \quad \text{thus} \quad \lim_{x \to 0} f(x) = 0
\]

(b)(ii) (6pts) No, \( f(x) \) is not continuous at \( x = 0 \). For \( x \neq 0 \), we have \( f(x) = x^2 + x \), and recall that polynomial functions are continuous and at \( x = 0 \) we have to check that \( \lim_{x \to 0^-} f(x) = f(0) \), note that
\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 + x = 0^2 + 0 = 0
\]
and so \( f(x) \) is not continuous for all real \( x \) and has a removable discontinuity at \( x = 0 \).

(c) (4pts) Choice (B). Discussion: Note that
\[
\lim_{x \to \infty} \frac{3x + 1}{\sqrt{8x^3} + 5} = \lim_{x \to \infty} \frac{x \cdot (3 + 1/x)}{\sqrt{8} + 5/x^3} = \frac{3\sqrt{8}}{2} \quad \text{and} \quad \lim_{x \to -\infty} \frac{3x + 1}{\sqrt{8x^3} + 5} = \frac{3}{2}
\]
thus we see that \( y = 3/2 \) is the only horizontal asymptote which implies choice (B).

3. (24pts) Start this problem on a new page. The following parts of this problem are not related.

(a) (12pts) Use the Squeeze Theorem to evaluate the following limit: \( \lim_{x \to 0} x^4 \cos \left( \frac{\pi - 4}{x^2} \right) \).

(b) (12pts) Find \( \lim_{x \to 0} \frac{\sin(3x)}{5x} \). Show all work and justify your answer.

Solution:

(a) (12pts) Note that, using that fact that \( x^4 \geq 0 \), we have
\[
-1 \leq \cos \left( \frac{\pi - 4}{x^2} \right) \leq 1 \quad \Rightarrow \quad -x^4 \leq x^4 \cos \left( \frac{\pi - 4}{x^2} \right) \leq x^4 \quad \Rightarrow \quad \lim_{x \to 0^-} -x^4 \leq \lim_{x \to 0} x^4 \cos \left( \frac{\pi - 4}{x^2} \right) \leq \lim_{x \to 0^+} x^4
\]
and since \( \lim_{x \to 0^-} -x^4 = \lim_{x \to 0^+} x^4 = 0 \), by the Squeeze theorem, we have \( \lim_{x \to 0} x^4 \cos \left( \frac{\pi - 4}{x^2} \right) = 0 \).

(b) (12pts) In this case, using the fact that \( \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1 \) we have
\[
\lim_{x \to 0} \frac{\sin(3x)}{5x} = \lim_{x \to 0} \frac{\sin(3x)}{x} \cdot \frac{1}{5} = \lim_{x \to 0} \frac{\sin(3x)}{3x} \cdot \frac{3}{5} = 1 \cdot \frac{3}{5} = \frac{3}{5}
\]

4. (24pts) Start this problem on a new page.

(a) (10pts) Use one-sided limits to find \( \lim_{x \to 3} \frac{|x - 3|}{x^3 - 3x^2} \). Show all work.

(b) (10pts) Suppose \( f(x) = \frac{|x - 3|}{x^3 - 3x^2} \). (i) (5pts) Find the limit \( \lim_{x \to 0} f(x) \). Show all work. (ii) (5pts) Is \( f(x) \) continuous for all real \( x \)? If not, classify the discontinuities of \( f(x) \). Explain.
The function \( g(x) = \begin{cases} 
\frac{|x-3|}{x^3 - 3x^2}, & \text{if } x > 2, \\
\frac{\sqrt{6x+1} - 5}{x-4}, & \text{if } x \leq 2, 
\end{cases} \)
has a \textit{vertical asymptote} at which choice below?

(No justification necessary - Choose only one answer, copy down the entire answer.)

(A) \( x = 0, x = 3 \) and \( x = 4 \)  (B) \( x = 0 \) (C) \( x = 3 \) and \( x = 4 \)  (D) \( x = 2 \) (E) None of these

Solution: (a)(10pts) Note that \( x > 3 \Rightarrow |x-3| = x-3 \) and \( x < 3 \Rightarrow |x-3| = -(x-3) \) thus
\[
\lim_{x \to 3^+} \frac{|x-3|}{x^3 - 3x^2} = \lim_{x \to 3^+} \frac{x-3}{x^2(x-3)} = \lim_{x \to 3^+} \frac{1}{x^2} = \frac{1}{9} \quad \text{and} \quad \lim_{x \to 3^-} \frac{|x-3|}{x^3 - 3x^2} = \lim_{x \to 3^-} \frac{-(x-3)}{x^2 \cdot (x-3)} = \lim_{x \to 3^-} \frac{-1}{x^2} = -\frac{1}{9}
\]
thus the two-sided limit at \( x = 3 \) does not exist, that is, \( \lim_{x \to 3} \frac{|x-3|}{x^3 - 3x^2} = \text{d.n.e.} \).

(b)(i)(5pts) Note that if \( x \to 0 \) then \( x \) is very close to 0 so we can assume \( x < 3 \) so \( |x-3| = -(x-3) \) thus
\[
\lim_{x \to 0} \frac{|x-3|}{x^3 - 3x^2} = \lim_{x \to 0} \frac{-(x-3)}{x^2 \cdot (x-3)} = \lim_{x \to 0} \frac{-1}{x^2} = \frac{-∞}{0} = -∞.
\]

(b)(ii)(5pts) No, the function is not defined at \( x = 0 \) and \( x = 3 \) and therefore not continuous for all real \( x \). From the work done in part (a), we see that \( f(x) \) has a \textit{jump discontinuity} at \( x = 3 \) and, from part (b)(i), we see that \( f(x) \) has a \textit{infinite discontinuity} at \( x = 0 \).

(c)(4pts) Choice (E). Discussion: Note that \( y = \frac{|x-3|}{x^3 - 3x^2} = \frac{|x-3|}{x^2(x-3)} \) is has no VAs for any \( x > 2 \) (there is a jump discontinuity at \( x = 3 \)) and \( y = \frac{\sqrt{6x+1} - 5}{x-4} \) is defined for all \( x \leq 2 \) thus the function \( g(x) \) has \textit{no} vertical asymptotes which implies choice (E).