1. (24pts) The following parts of this problem are not related.

(a)(12pts) Given the functions $f(x) = \dfrac{1}{x^2 - 3}$ and $g(x) = \sqrt{x+1}$, find the composition $(f \circ g)(x)$ and state the domain using interval notation.

(b)(12pts) Evaluate the limit: $\lim\limits_{x \to 1} \dfrac{x^2 + 4x - 5}{x^2 - x}$. Show all work.

Solution: (a)(12pts) Note that the composition is

$$(f \circ g)(x) = f(g(x)) = \frac{1}{(\sqrt{x+1})^2 - 3} = \frac{1}{x+1-3} = \frac{1}{x-2}$$

and for $f(g(x))$ to be well defined we need $x \neq 2$ and the domain of $g(x)$ is $[-1, \infty)$, so the domain of $f(g(x))$ in interval notation is $[-1, 2) \cup (2, \infty)$.

(b)(12pts) Applying the limit directly gives a "0/0" type indeterminate form and factoring yields

$$\lim_{x \to 1} \frac{x^2 + 4x - 5}{x^2 - x} \overset{0/0}{=} \lim_{x \to 1} \frac{(x-1)(x+5)}{x(x-1)} = \lim_{x \to 1} \frac{\cancel{(x-1)} \cdot (x+5)}{x \cdot \cancel{(x-1)}} = \lim_{x \to 1} \frac{x+5}{x} = \frac{1+5}{1} = 6.$$

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2. (28pts) Start this problem on a **new** page. The following parts of this problem are not related.

(a)(12pts) Evaluate the limit: $\lim\limits_{x \to 4} \dfrac{\sqrt{6x+1} - 5}{x - 4}$. Show all work.

(b)(12pts) Suppose $f(x) = \begin{cases} x^2 + x, & \text{if } x \neq 0 \\ \cos(x), & \text{if } x = 0 \end{cases}$. (i)(6pts) Find the $\lim\limits_{x \to 0} f(x)$. (ii)(6pts) Is $f(x)$ continuous for all real x? If not, classify the discontinuities of $f(x)$. Be sure to show that all three conditions of continuity have been satisfied and justify your answer.

(c)(4pts) The function $f(x) = \dfrac{3x+1}{\sqrt[3]{8x^3 + 5}}$ has a *horizontal asymptote* at which choice below? (**No justification necessary** - *Choose only* <u>*one*</u> *answer, copy down the entire answer.*)

(A) $y = 0$ (B) $y = \dfrac{3}{2}$ (C) $y = 0$ and $y = 3/2$ (D) $y = -3/2$ and $y = 3/2$ (E) None of these

Solution:

(a)(12pts) Multiplying by the conjugate yields:

$$\lim_{x \to 4} \frac{\sqrt{6x+1} - 5}{x - 4} \overset{\text{“0/0”}}{=} \lim_{x \to 4} \frac{\sqrt{6x+1} - 5}{x - 4} \cdot \frac{\sqrt{6x+1} + 5}{\sqrt{6x+1} + 5}$$

$$= \lim_{x \to 4} \frac{(6x+1) - 25}{(x-4)[\sqrt{6x+1} + 5]} = \lim_{x \to 4} \frac{6\cancel{(x-4)}}{\cancel{(x-4)}[\sqrt{6x+1} + 5]} = 6/10 = \boxed{3/5.}$$

(b)(i)(6pts) Note that

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 + x = 0 \text{ and } \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 + x = 0 \text{ thus } \boxed{\lim_{x\to 0} f(x) = 0}$$

(b)(ii)(6pts) No, $f(x)$ is not continuous at $x = 0$. For $x \neq 0$, we have $f(x) = x^2 + x$, and recall that polynomial are continuous and at $x = 0$ we have to check that $\lim_{x\to 0} f(x) = f(0)$, note that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 + x = 0^2 + 0 = 0 \text{ and } f(0) = \cos(0) = 1 \Rightarrow \lim_{x\to 0} f(x) \neq f(0)$$

and so $\boxed{f(x) \text{ is not continuous for all real } x \text{ and has a } \textit{removable discontinuity} \text{ at } x = 0.}$

(c)(4pts) $\boxed{\text{Choice (B).}}$ <u>Discussion:</u> Note that

$$\lim_{x\to\infty} \frac{3x+1}{\sqrt[3]{8x^3+5}} = \lim_{x\to\infty} \frac{x\cdot(3+1/x)}{x\cdot\sqrt[3]{8+5/x^3}} = \frac{3}{\sqrt[3]{8}} = \frac{3}{2} \text{ and } \lim_{x\to-\infty} \frac{3x+1}{\sqrt[3]{8x^3+5}} \overset{DOP}{=} \frac{3}{2}$$

thus we see that $y = 3/2$ is the only horizontal asymptote which implies choice (B).

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3. (24pts) Start this problem on a **new** page. The following parts of this problem are not related.

(a)(12pts) Use the Squeeze Theorem to evaluate the following limit: $\lim_{x\to 0} x^4 \cos\left(\frac{\pi - 4}{x^2}\right)$.

(b)(12pts) Find $\lim_{x\to 0} \frac{\sin(3x)}{5x}$. Show all work and justify your answer.

Solution:

(a)(12pts) Note that, using that fact that $x^4 \geq 0$, we have

$$-1 \leq \cos\left(\frac{\pi-4}{x^2}\right) \leq 1 \;\Rightarrow\; -x^4 \leq x^4\cos\left(\frac{\pi-4}{x^2}\right) \leq x^4 \;\Rightarrow\; \lim_{x\to 0} -x^4 \leq \lim_{x\to 0} x^4 \cos\left(\frac{\pi-4}{x^2}\right) \leq \lim_{x\to 0} x^4$$

and since $\lim_{x\to 0} -x^4 = \lim_{x\to 0} x^4 = 0$, by the Squeeze theorem, we have $\lim_{x\to 0} x^4 \cos\left(\frac{\pi-4}{x^2}\right) = 0$.

(b)(12pts) In this case, using the fact that $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$ we have

$$\lim_{x\to 0} \frac{\sin(3x)}{5x} = \lim_{x\to 0} \frac{\sin(3x)}{x} \cdot \frac{1}{5} = \lim_{x\to 0} \frac{\sin(3x)}{3x} \cdot \frac{3}{5} = 1 \cdot \frac{3}{5} = \frac{3}{5}.$$

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4. (24pts) Start this problem on a **new** page.

(a)(10pts) Use one-sided limits to find $\lim_{x\to 3} \frac{|x-3|}{x^3 - 3x^2}$. Show all work.

(b)(10pts) Suppose $f(x) = \frac{|x-3|}{x^3 - 3x^2}$. ($i$)(5pts) Find the limit $\lim_{x\to 0} f(x)$. Show all work.(ii)(5pts) Is $f(x)$ continuous for all real x? If not, classify the discontinuities of $f(x)$. Explain.

(c)(4pts) The function $g(x) = \begin{cases} \dfrac{|x-3|}{x^3-3x^2}, & \text{if } x > 2, \\ \dfrac{\sqrt{6x+1}-5}{x-4}, & \text{if } x \le 2, \end{cases}$ has a *vertical asymptote* at which choice below?

(**No justification necessary** - *Choose only one answer, copy down the entire answer.*)

(A) $x=0, x=3$ and $x=4$ (B) $x=0$ (C) $x=3$ and $x=4$ (D) $x=2$ (E) None of these

Solution: (a)(10pts) Note that $x > 3 \Rightarrow |x-3| = x-3$ and $x < 3 \Rightarrow |x-3| = -(x-3)$ thus

$$\lim_{x\to 3^+} \frac{|x-3|}{x^3-3x^2} = \lim_{x\to 3^+} \frac{x-3}{x^2(x-3)} = \lim_{x\to 3^+} \frac{1}{x^2} = \frac{1}{9} \quad \text{and} \quad \lim_{x\to 3^-} \frac{|x-3|}{x^3-3x^2} = \lim_{x\to 3^+} \frac{-(x-3)}{x^2\cdot(x-3)} = -\frac{1}{9}$$

thus the two-sided limit at $x=3$ does not exist, that is, $\lim_{x\to 3} \frac{|x-3|}{x^3-3x^2} = \text{d.n.e.}$

(b)(i)(5pts) Note that if $x \to 0$ then x is very close to 0 so we can assume $x < 3$ so $|x-3| = -(x-3)$ thus

$$\lim_{x\to 0} \frac{|x-3|}{x^3-3x^2} = \lim_{x\to 0} \frac{-(x-3)}{x^2\cdot(x-3)} = \lim_{x\to 0} -\frac{1}{x^2} \overset{N/0}{=} -\infty.$$

(b)(ii)(5pts) **No**, the function is not defined at $x=0$ and $x=3$ and therefore not continuous for all real x. From the work done in part (a), we see that $f(x)$ has a **jump discontinuity** at $x=3$ and, from part (b)(i), we see that $f(x)$ has a **infinite discontinuity** at $x=0$.

(c)(4pts) $\boxed{\text{Choice (E).}}$ Discussion: Note that $y = \frac{|x-3|}{x^3-3x^2} = \frac{|x-3|}{x^2(x-3)}$ is has no VAs for any $x > 2$ (there is a jump discontinuity at $x=3$) and $y = \frac{\sqrt{6x+1}-5}{x-4}$ is defined for all $x \le 2$ thus the function $g(x)$ has no vertical asymptotes which implies choice (E).