1. (30 pts) Evaluate the following

 (a) $\displaystyle\int_0^1 \frac{3}{9+x^2}\,dx$

 (b) $\displaystyle\int x\cosh(x^2)\,dx$

 (c) $\displaystyle\int_0^{\ln(\sqrt{3})} \frac{e^{2x}}{\sqrt{3+e^{2x}}}\,dx$

Solution:

(a) Factor a 9 out the denominator

$$\int_0^1 \frac{1}{1+\left(\frac{x}{3}\right)^2}\left(\frac{1}{3}\right)\,dx$$

Let $u=\frac{1}{3}x$ so $du=\frac{1}{3}dx$. The integration bounds change as follows:

$$x=0 \quad u=0$$
$$x=1 \quad u=1/3$$

we then have

$$\int_0^{1/3}\frac{1}{1+u^2}\,du = \tan^{-1}u\Big|_0^{1/3} = \tan^{-1}\left(\frac{1}{3}\right)$$

(b) Let $u=x^2$, $du=2xdx$. The integral becomes

$$\int \frac{1}{2}\cosh(u)du = \frac{1}{2}\sinh(u)+C = \frac{1}{2}\sinh(x^2)+C$$

(c) Let $u=3+e^{2x}$ so $du=2e^{2x}$. The integration bounds are $u=4$ and $u=3+e^{2\ln(\sqrt{3})}=3+e^{\ln(3)}=6$

$$\int_0^{\ln(\sqrt{3})}\frac{e^{2x}}{\sqrt{3+e^{2x}}}\,dx = \int_4^6 \frac{1}{\sqrt{u}}\left(\frac{1}{2}du\right) = \left[\sqrt{u}\right]_4^6 = \sqrt{6}-\sqrt{4} = \sqrt{6}-2$$

2. (25 pts) Compute the following limits

 (a) $\displaystyle\lim_{x\to-\infty}\frac{x^2}{e^{1-x}}$

(b) $\lim\limits_{x \to 0} \arctan(x) \cos\left(\dfrac{2}{x}\right)$ (hint: use the squeeze theorem)

(c) $\lim\limits_{\underbrace{x}_{\text{typo}} \to -1^-} \dfrac{t+1}{|t+1|}$

Solution:

(a) The function is indeterminant of type $\frac{\infty}{\infty}$ so we can use L'Hôpital's rule.

$$\begin{aligned} &\lim_{x \to -\infty} \frac{x^2}{e^{1-x}} \\ \overset{\text{L'H}}{=} &\lim_{x \to -\infty} \frac{2x}{-e^{1-x}} \quad \left(\text{indeterminant type } \frac{\infty}{\infty}\right) \\ \overset{\text{L'H}}{=} &\lim_{x \to -\infty} \frac{2}{e^{1-x}} = 0 \end{aligned}$$

(b) Using the squeeze theorem based on the hint we note that for $x \neq 0$

$$-1 \leq \cos(2/x) \leq 1$$
$$-\arctan(x) \leq \arctan(x)\cos(2/x) \leq \arctan(x)$$

and

$$\lim_{x \to 0} \arctan(x) = 0$$
$$\lim_{x \to 0} -\arctan(x) = 0$$

Then by the squeeze theorem

$$\lim_{x \to 0} \arctan(x) \cos\left(\frac{2}{x}\right) = 0$$

(c)

$$|t+1| = \begin{cases} t+1 & t \geq -1 \\ -t-1 & t < -1 \end{cases}$$

$$\begin{aligned} \lim_{t \to -1^-} \frac{t+1}{|t+1|} &= \lim_{t \to -1^-} \frac{t+1}{-t-1} \\ &= \lim_{t \to -1^-} (-1) \\ &= -1 \end{aligned}$$

3. (25 pts) The following problems are unrelated

(a) Find y' for $y = \sqrt{x}^{\sqrt{x}} e^{x^2}$.

(b) Find the tangent line to $f(x) = \ln(x) \log_2(x)$ at $x = 2$.

(c) Approximate the value of $\displaystyle\int_0^1 3\sin^{-1}(x)dx$ using R_2, the right-endpoint approximation with two equal subintervals. Simplify your answer.

Solution:

(a) Using logarithmic differentiation

$$\ln(y) = \ln\left(\sqrt{x}^{\sqrt{x}} e^{x^2}\right)$$
$$= \frac{1}{2}\sqrt{x}\ln(x) + x^2$$

Differentiating both sides

$$\frac{1}{y}y' = \frac{1}{4}x^{-1/2}\ln(x) + \frac{1}{2}\sqrt{x}\left(\frac{1}{x}\right) + 2x = \frac{1}{4\sqrt{x}}\ln(x) + \frac{1}{2\sqrt{x}} + 2x$$

Therefore

$$y' = y\left(\frac{1}{4\sqrt{x}}\ln(x) + \frac{1}{2\sqrt{x}} + 2x\right) = x^{\sqrt{x}}e^{x^2}\left(\frac{1}{4\sqrt{x}}\ln(x) + \frac{1}{2\sqrt{x}} + 2x\right)$$

An alternative approach is to write the function as

$$y = e^{\ln(\sqrt{x}^{\sqrt{x}}e^{x^2})} = e^{\frac{1}{2}\sqrt{x}\ln(x)+2x}$$

and then differentiate explicitly.

(b) The derivative is

$$f'(x) = \frac{1}{x}\log_2(x) + \ln(x)\left(\frac{1}{\ln(2)x}\right)$$

at $x = 2$ the slope of the tangent line is

$$f'(2) = \frac{1}{2}\log_2(2) + \ln(2)\left(\frac{1}{2\ln(2)}\right) = 1$$

The tangent line passes through the y-coordinate

$$f(2) = \ln(2)\log_2(2) = \ln(2)$$

The equation to the tangent line in point-slope form is

$$y - \ln(2) = (x - 2)$$

(c) Let $f(x) = 3\sin^{-1}(x)$. with R_2 on the given interval, $b = 1$, $a = 0$. Therefore

$$\Delta x = \frac{1}{2} \qquad x_i = \frac{i}{2}$$

$$\int_0^1 3\sin^{-1}(x)dx \approx R_2 = \frac{1}{2}f(\frac{1}{2}) + \frac{1}{2}f(1)$$
$$= \frac{3}{2}\sin^{-1}\left(\frac{1}{2}\right) + \frac{3}{2}\sin^{-1}(1)$$
$$= \frac{3}{2}\left(\frac{\pi}{6} + \frac{\pi}{2}\right) = \boxed{\pi}$$

4. (15 pts) In this problem we consider the function $f(x) = 3xe^{2x-10}$ at $x = 5$.

(a) Find the linearization of $f(x)$ at $x = 5$.

(b) Find the interval(s) on which the function is increasing.

Solution:

(a) We want the linearization at $a = 5$. We'll need the derivative

$$f'(x) = 3e^{2x-10} + 6xe^{2x-10}$$

We then compute

$$f(a) = 3(5)e^{10-10} = 15$$
$$f'(a) = 3e^0 + 6(5)e^0 = 33.$$

The linearization is then

$$L(x) = 15 + 33(x - 5)$$

(b) From part (a), $f'(x) = (3 + 6x)e^{2x-10}$. Since the exponential is always positive, then $f'(x) > 0$ whenever

$$3 + 6x > 0 \implies x > -\frac{1}{2}$$

The interval where $f'(x) > 0$ is therefore $(-1/2, \infty)$.

5. (20 pts) A colony of bacteria is growing exponentially. If there are 200 bacteria after 2 hours and 800 bacteria after 5 hours. Find the growth rate k.

Solution: Since the bacteria are following an exponential growth trend, they are described by the function

$$y(t) = y_0 e^{kt}$$

Using the information given, $y(2) = 200$ and $y(5) = 800$ so we have

$$200 = y_0 e^{2k} \implies 1 = \frac{y_0}{200} e^{2k}$$
$$800 = y_0 e^{5k} \implies 1 = \frac{y_0}{800} e^{5k}$$

We can now equate these

$$\frac{y_0}{200} e^{2k} = \frac{y_0}{800} e^{5k}$$
$$\implies \frac{800}{200} = \frac{e^{5k}}{e^{2k}}$$
$$\implies 4 = e^{3k}$$

so $k = \frac{1}{3} \ln(4)$

6. (15 pts) A boat leaves a dock traveling due south at 20 km/hour. At the same time, a boat 20 km due west from the dock begins traveling east at 10 km/hour. At what time is the distance between the two boats a minimum? (hint: write the coordinate of each boat as a function of time)

Solution: Let the dock be at the origin. The y coordinate of the boat traveling south is $y(t) = 0 - 20t$. The x coordinate of the second boat is then $x(t) = -20 + 10t$. The distance between the two boats is then

$$\begin{aligned} D(t) &= \sqrt{x(t)^2 + y(t)^2} \\ &= \sqrt{(-20+10t)^2 + (-20t)^2} \\ &= \sqrt{400 - 400t + 500t^2} \end{aligned}$$

Rather than maximizing the distance, it's much easier to maximize the distance squared

$$D(t)^2 = f(t) = 400 - 400t + 500t^2$$

Then

$$f'(t) = -400 + 1000t$$

So we have a critical value at $t = \frac{400}{1000} = \frac{2}{5}$. We verify this is a minimum by the second derivative test where

$$f''(t) = 1000 > 0$$

so $t = \frac{2}{5}$ gives a minimum.

7. (20 pts) The following problems consider a function $f(x)$ that is continuous everywhere with the following properties

- $f'(x) > 0$ on $(-\infty, 1)$, $f'(x) < 0$ on $(1, \infty)$ and $f'(1) = 0$.
- $f''(x) > 0$ on $\underbrace{(\infty, -2)}_{\text{typo}}$ and $(2, \infty)$, $f''(x) < 0$ on $(-2, 2)$.
- $\lim_{x \to -\infty} f(x) = -\infty$.
- $\lim_{x \to \infty} f(x) = 1$.
- $f(-3) = -2$
- $f(1) = 2$

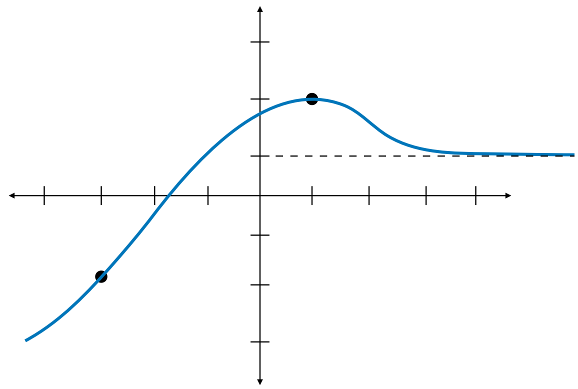
(a) Sketch a single curve with all of the above properties.

(b) Can you guarantee that $f(x)$ has a local maximum? You must fully justify your answer, referencing your graph from part (a) is not sufficient.

(c) Can you guarantee that there is a value of c such that $f(c) = 0$? Fully justify your answer.

Solution:

(a) One possible such curve is the following



(b) We know that $f'(1) = 0$ so $x = 1$ gives a critical point. At this value $f''(1) < 0$ so we can guarantee that $f(1)$ is a local maximum.

We can also use the fact that $f'(x) > 0$ to the left of $x = 1$ and $f'(x) < 0$ to the right of $x = 1$ to make the same conclusion.

(c) Since $f(-3) = -2$ and $f(1) = 2$ and $-3 \le 0 \le 2$ and $f(x)$ is continuous then the intermediate value theorem (IVT) guarantees that there exists a point c in the interval $(-3, 2)$ such that $f(c) = 0$.