1. (25 pts) Compute the following

(a) $\displaystyle\int_0^3 \sqrt{9-x^2}dx$ (Hint: Use geometry)

(b) $\displaystyle\int_{-\pi}^{\pi/2} \sin(x)\sin(\cos(x))dx$

(c) $\displaystyle\frac{d}{dx}\int_1^{x^2} (t^3-4t)dt$

(d) $\displaystyle\int_{-\pi/2}^{\pi/2} x\sin^2(x^8)\cos(x^8)dx$

Solution:

(a) Note that this is area of a quarter circle centered at the origin with radius 3. So the integral evaluates to

$$\int_0^3 \sqrt{9-x^2}dx = \frac{1}{4}\pi(3)^2 = \boxed{9/4\pi}$$

(b) Let $u=\cos(x)$ so that $du=-\sin(x)dx$. The integration bounds here become $x=-\pi \implies u=-1$ and $x=\pi/2 \implies u=0$. The integral in u is therefore

$$\begin{aligned}\int_{-1}^{0}(-\sin(u))du &= \cos(u)\Big|_{-1}^{0} \\ &= \cos(0)-\cos(-1) \\ &= \boxed{1-\cos(-1) = 1-\cos(1)}\end{aligned}$$

(c) Using part 2 of the fundamental theorem of calculus,

$$\begin{aligned}\int_1^{x^2}(t^3-4t)dt &= \left[\frac{1}{4}t^4-2t^2\right]_1^{x^2} \\ &= \frac{1}{4}x^8-2x^4+\frac{7}{4}\end{aligned}$$

So now we compute

$$\begin{aligned}\frac{d}{dx}\int_1^{x^2}(t^3-4t)dt &= \frac{d}{dx}\left[\frac{1}{4}x^8-2x^2+\frac{7}{4}\right] \\ &= \boxed{2x^7-8x^3}\end{aligned}$$

(d) If we let $f(x)=x\sin^2(x^3)\cos(x^3)$ then we find

$$\begin{aligned}f(-x) &= -x\sin^2\left((-x)^3\right)\cos\left((-x)^3\right) \\ &= -x\sin^2(-x^3)\cos(-x^3) \\ &= -x\sin^2(x^3)\cos(x^3) = -f(x)\end{aligned}$$

so the function within the integral is odd. Since the interval is symmetric about the origin then

$$\int_{-\pi/2}^{\pi/2} x\sin^2(x^3)\cos(x^3)dx = 0$$

2. (15 pts) The following problems are unrelated.

(a) Find the slant asymptote of $f(x) = \dfrac{x^2+6x-4}{3x-6}$.

(b) Find the value(s) c such that $f(c) = f_{ave}$ given $f(x) = -x^2+2x+2$ on the interval $[0,3]$

Solution:

(a) The slant asymptote can be found via polynomial division

$$\begin{array}{r|l} & \tfrac{1}{3}x + \tfrac{8}{3} \\ \hline 3x-6 & x^2+6x-4 \\ & -x^2+2x \\ \hline & 8x-4 \\ & -8x+16 \\ \hline & 12 \end{array}$$

Therefore, $f(x)$ can be rewritten

$$f(x) = \frac{x^2+6x-4}{3x-6} = \frac{1}{3}x + \frac{8}{3}x + \frac{12}{3x-6}$$

The slant asymptote is then given by

$$\boxed{y = \frac{1}{3}x + \frac{8}{3}}$$

(b) Such values of c is guaranteed by the mean value theorem for integrals.

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{3-0}\int_0^3 (-x^2+2x+2)dx \\ &= \frac{1}{3}\left[-\frac{1}{3}x^3 + x^2 + 2x\right]_0^3 \\ &= \frac{1}{3}\left(-\frac{1}{3}(3)^3 + (3)^2 + 2(3)\right) \\ &= 2 \end{aligned}$$

The value of c is given by

$$\begin{aligned} -c^2+2c+2 &= 2 \\ \implies -c^2+2c &= 0 \\ \implies \boxed{c=0, \quad c=2} \end{aligned}$$

3. (25 pts) The following problems are unrelated.

(a) We wish to design a box with a square base and a surface area of 108 square inches. What dimensions will produce a box with maximum volume?

(b) Use one step of Newton's method to approximate the location of the local minimum to $f(x) = (x+1)^4 - 32(x+1)$ with $x_0 = 0$.

(c) A car is initially moving at a velocity of 20 meters/s and begins to decelerate $a(t) = -3\sqrt{t}$ meters/s^2. How far does the car move after 4 seconds relative to its position at $t = 0$?

Solution:

(a) The base of the box has length x and the height of the box is y. The surface area is the sum of the 4 sides plus the top and bottom

$$2x^2 + 4xy = 108 \implies y = \frac{108 - x^2}{4x}$$

The volume of the box is given by

$$\begin{aligned} V = x^2 y &= \frac{x^2(108 - 2x^2)}{4x} \\ &= 27x - \frac{1}{2}x^3 \end{aligned}$$

Differentiating and setting the derivative to zero we find

$$\begin{aligned} V' &= 27 - \frac{3}{2}x^2 = 0 \\ \implies x &= \sqrt{18} = 3\sqrt{2} \end{aligned}$$

is the only physical critical point. Here, $V'' = -3x$ which is negative at $x = 3\sqrt{2}$ so the value gives a maximum volume.

The height of the box is $y = \frac{108 - 2(3\sqrt{2})^2}{12\sqrt{2}} = \frac{6}{\sqrt{2}} = 3\sqrt{2}$. The dimensions of the box are then

$$\boxed{3\sqrt{2} \text{ in} \times 3\sqrt{2} \text{ in} \times 3\sqrt{2} \text{ in}}$$

(b) This problem amounts to using Newton's method to find a root of the derivative $f'(x) = 4(x+1)^3 - 32$ (we also need the second derivative $f''(x) = 12(x+1)^2$). The first approximation to the root using Newton's method is

$$\begin{aligned} x_1 &= x_0 - \frac{f'(x_0)}{f''(x_0)} \\ &= 0 - \frac{4 - 32}{12} \\ &= \frac{28}{12} \\ &= \boxed{\frac{7}{3}} \end{aligned}$$

(c) This is the initial value problem

$$x''(t) = -3\sqrt{t}, \quad x'(0) = 20, \quad x(0) = 0$$

So the velocity is

$$x'(t) = C - 2t^{3/2}$$

Since $x'(0) = 20$, $C = 20$ so the velocity $x'(t) = 20 - 2\sqrt{t}$. We find the position by integrating once more

$$x(t) = 20t - \frac{4}{5}t^{5/2} + D$$

The initial condition $x(0) = 0$ gives $D = 0$. Therefore the position of the car relative to the initial condition is

$$x(t) = 20t - \frac{4}{5}t^{5/2},$$

so that after 4 seconds,

$$\begin{aligned} x(2) &= 80 - \frac{4}{5}4^{5/2} \\ &= \boxed{80 - \frac{4(32)}{5}} \end{aligned}$$

4. (15 pts) In this problem, define the function $g(x) = \int_2^x \frac{4}{t} dt$.

(a) Compute $g(2)$ and $g'(2)$.

(b) Show that $g'(x) = -\frac{d}{dx}\left[g\left(\frac{1}{x}\right)\right]$

Solution:

(a)

$$g(2) = \int_2^2 \frac{4}{t} dt = \boxed{0}$$

$$g'(2) = \left.\frac{4}{x}\right|_{x=2} = \frac{4}{2} = \boxed{2}$$

(b) Using FTCI, we have $f'(x) = \frac{4}{x}$ and

$$\begin{aligned} g'(1/x) &= \frac{d}{dx} \int_2^{1/x} \frac{4}{t} dt \\ (\text{let } u = 1/x) \implies &= \frac{d}{dx} \int_2^u \frac{4}{t} dt \\ &= \frac{df}{du} \cdot \frac{du}{dx} \\ &= \frac{4}{\left(\frac{1}{x}\right)} \left(-\frac{1}{x^2}\right) \\ &= -\frac{4}{x} \end{aligned}$$

Therefore we have shown $g'(x) = -\frac{d}{dx}\left[g\left(\frac{1}{x}\right)\right]$

5. (20 pts) The following problems are unrelated.

(a) Consider the definite integral $\int_1^4 x^2 - 2x\, dx$

(i) Approximate the integral with R_3, the right Riemann sum consisting of three terms.

(ii) Write down the right Riemann sum with n terms, R_n. Use the formulas provided to evaluate the Riemann sum.

(b) Compute $\lim_{n\to\infty} \sum_{i=1}^{n} \left[3\left(\frac{2i}{n}\right)^3 - \left(\frac{2i}{n}\right)^2 + 2\right] \frac{2}{n}$ by evaluating a definite integral.

Solution:

(a) In this solution, we use $f(x) = x^2 - 2x$

(i) Using $n = 3$, we have $\Delta x = 1$ and $x_1 = 2$, $x_2 = 3$, $x_3 = 4$

$$\begin{aligned} R_3 &= f(2) + f(3) + f(4) \\ &= \left(2^2 - 2(2)\right) + \left(3^2 - 2(3)\right) + \left(4^2 - 2(4)\right) \\ &= 0 + 3 + 8 \\ &= \boxed{11} \end{aligned}$$

(ii) The Riemann sum using n terms uses $\Delta x = \frac{b-a}{n} = \frac{3}{n}$ and $x_i = 1 + \frac{3}{n}i$ So

$$\begin{aligned} R_n &= \sum_{i=1}^{n} f(x_i)\Delta x \\ &= \sum_{i=1}^{n} \left[\left(1 + \frac{3i}{n}\right)^2 - 2\left(1 + \frac{3i}{n}\right)\right] \frac{3}{n} \\ &= \sum_{i=1}^{n} \left[-1 + \frac{9i^2}{n^2}\right] \frac{3}{n} \\ &= \frac{3}{n}\left[-n + \frac{9}{n^2} \cdot \frac{n(n+1)(2n+1)}{6}\right] \\ &= -3 + \frac{9n(n+1)(2n+1)}{2n^3} \end{aligned}$$

(b) This is a Riemann sum with $\Delta x = \frac{2}{n}$ and $x_i = \frac{2i}{n}$ so that the integration bounds are $a = 0$ and $b = 2$. The definite integral corresponding to the Riemann sum is

$$\begin{aligned} \int_0^2 3x^3 - x^2 + 2dx &= \left[\frac{3}{4}x^4 - \frac{1}{3}x^3 + 2x\right]_0^2 \\ &= \frac{3}{4}(16) - \frac{1}{3}(8) + 4 \\ &= 12 - \frac{8}{3} + 4 \\ &= \boxed{16 - \frac{8}{3} = \frac{40}{3}} \end{aligned}$$