1. (20 pts) Find the requested derivative for each of the following.

   (a) Find y' for $y = \sqrt[4]{x}\cos(x)$.

   (b) Find $\dfrac{dy}{dx}$ for $(x+y)^3 = x^3 + y^3$.

   (c) Find $\dfrac{d^2y}{dt^2}$ for $y = \sec(t)$.

   (d) Find $f'(1)$ for $f(x) = \dfrac{x^2-4}{x-3}$.

Solution:

(a) Here we need to utilize the product rule

$$y' = \frac{d}{dx}\left[x^{1/4}\right]\cos(x) + x^{1/4}\frac{d}{dx}\left[\cos(x)\right] = \boxed{\frac{1}{4}x^{-3/4}\cos(x) - x^{1/4}\sin(x)}$$

(b) Using implicit differentiation,

$$3(x+y)^2\left(1+\frac{dy}{dx}\right) = 3x^2 + 3y^2\frac{dy}{dx}$$

$$\implies \frac{dy}{dx} = \frac{3x^2 - 3(x+y)^2}{3(x+y)^2 - 3y^2} = \boxed{\frac{-2xy - y^2}{x^2 + 2xy}}$$

(c) We first compute $\frac{dy}{dt}$ and utilize the chain rule

$$\frac{d}{dt}\left[\sec(t)\right] = \sec(t)\tan(t)$$

The second derivative is found via

$$\begin{aligned}\frac{d^2}{dt^2}\left[\sec(t)\right] &= \frac{d}{dt}\left[\sec(t)\tan(t)\right]\\ &= \left[\sec(t)\right]'\tan(t) + \sec(t)\left[\tan(t)\right]'\\ &= \sec(t)\tan^2(t) + \sec^3(t)\end{aligned}$$

(d) The first derivative is

$$\begin{aligned}f'(x) &= \frac{2x(x-3) - (x^2-4)(1)}{(x-3)^2}\\ &= \frac{x^2 - 6x + 4}{(x-3)^2}\end{aligned}$$

Therefore $f'(1) = \frac{-1}{(-2)^2} = \boxed{-\frac{1}{4}}$

2. (16 pts) The following problems are unrelated

(a) Use the definition of a derivative to find the derivative of $f(t) = 10 - t^2$.

(b) Find the linearization of $\tan(x - \pi/4)$ at $a = \pi/2$ and use it to approximate the value of $\tan(\pi/5)$.

Solution:

(a)

$$\begin{aligned} f'(t) &= \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} \\ &= \lim_{h \to 0} \frac{10 - (t+h)^2 - (10 - t^2)}{h} \\ &= \lim_{h \to 0} \frac{10 - t^2 - 2th - h^2 - 10 + t^2}{h} \\ &= \lim_{h \to 0} \frac{h(-2t - h)}{h} \\ &= \boxed{-2t} \end{aligned}$$

(b) The point given to compute the linearization is $a = \pi/2$, where at this point

$$\begin{aligned} f(a) &= \tan(\pi/4) = 1 \\ f'(a) &= \sec^2(\pi/4) = \left(\frac{2}{\sqrt{2}}\right)^2 = \frac{4}{2} = 2 \end{aligned}$$

Using the linearization formula,

$$L(x) = 1 + 2(x - \pi/2)$$

To approximate the value of $\tan(\frac{\pi}{5})$, we evaluate the linearization at $x = \frac{9\pi}{20}$ and find

$$\begin{aligned} L\left(\frac{9}{20}\pi\right) &= 1 + 2\left(\frac{9\pi}{20} - \frac{\pi}{2}\right) \\ &= 1 - \frac{\pi}{10} \end{aligned}$$

3. (24 pts) The position of a particle along a straight line is described by the function

$$s(t) = \frac{t}{4} - \frac{1}{2}\sin(t),$$

on the interval $[0, 2\pi]$.

(a) What is the velocity of the object at any time t?

(b) What is the acceleration of the object at any time t?

(c) Find all critical values of $s(t)$ on the interval $(0, 2\pi)$.

(d) Use the second derivative test to classify the local extrema of $s(t)$ on the interval $(0, 2\pi)$.

(e) Find the absolute maximum and minimum values of the **velocity** on the interval $[0, 2\pi]$.

Solution:

(a) velocity is the derivative:

$$s'(t) = v(t) = \frac{1}{4} - \frac{\cos(t)}{2}$$

(b) The acceleration is the second derivative

$$s''(t) = a(t) = \frac{\sin(t)}{2}$$

(c) Critical values occur when $s'(t) = 0$, which occur when

$$\cos(t) = \frac{1}{2} \implies t = \frac{\pi}{3}, \frac{5\pi}{3}$$

(d) Using the second derivative test, we evaluate $s''(t)$ at the appropriate critical points found in the previous part.

At $t = \pi/3$, $s(\pi/3) = \frac{\pi}{12} - \frac{\sqrt{3}}{4}$. The second derivative evaluated at this point is

$$s''(\pi/3) = \frac{\sin(\pi/3)}{2} = \frac{\sqrt{3}}{4} > 0$$

so the point $\left(\frac{\pi}{3}, \frac{\pi}{12} - \frac{\sqrt{3}}{4}\right)$ is a **local minimum**.

At $t = 5\pi/3$, $s(5\pi/3) = \frac{5\pi}{12} + \frac{\sqrt{3}}{4}$ The second derivative evaluated at this point is

$$s''(5\pi/3) = \frac{\sin(\pi/3)}{2} = -\frac{\sqrt{3}}{4} < 0$$

Therefore, at the point $\left(\frac{5\pi}{3}, \frac{5\pi}{12} + \frac{\sqrt{3}}{4}\right)$ is a **local maximum**.

(e) We can find the point at which the velocity is maximum by finding points at which the acceleration is 0 on the interval $(0, 2\pi)$

$$\frac{\sin t}{2} = 0 \implies t = \pi.$$

We also must check the end points of the interval and can find the maximum values of the velocity Therefore the absolute maximum value of the velocity is $\boxed{\frac{3}{4}}$ and the absolute

t value	$v(t)$	Classification
0	$-\frac{1}{4}$	Absolute min
π	$\frac{3}{4}$	Absolute max
2π	$-\frac{1}{4}$	Absolute min

minimum velocity is $\boxed{-1/4}$.

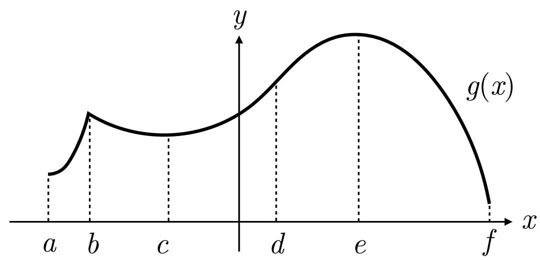
4. (20 pts) The following problems are unrelated.

(a) (i) State Rolle's theorem.

(ii) Verify that $f(x) = x^2 - 2x - 8$ satisfies the assumptions of Rolle's theorem on the interval $[-1, 3]$. Then find all values of c whose existence is guaranteed by Rolle's theorem on the interval.

(b) Below is a sketch of a function $g(x)$ on the interval $[a, f]$. Use the values of the x-coordinates specified to identify the following. No justification is necessary.

(i) Critical numbers of $g(x)$.
(ii) Locations of the absolute extrema of $g(x)$.
(iii) Intervals where $g'(x) \geq 0$.
(iv) Intervals where $g''(x) > 0$.



Solution:

(a) (i) Let $f(x)$ be a function that satisfies the following

a. f is continuous on the closed interval $[a, b]$
b. f is is differentiable on the open interval (a, b)
c. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

(ii) Rolle's can be used becasue the function is both continuous and differentiable as a polynomial and $f(-1) = f(3) = -5$.
To find c we take the derivative $f'(x) = 2x - 2$ and solve for $f'(c) = 0$. We find that $c = 1$, which is in our interval.

(b) (i) Critical numbers occur at $x = b, c, e$

(ii) The absolute maximum of $g(x)$ occurs when $x = e$. The absolute minimum occurs when $x = f$.

(iii) The function is nondecreasing ($g'(x) \geq 0$) on the intervals (a, b) and $[c, e]$

(iv) The function is concave up on (a, b) and (b, d).

5. (20 pts) A thin sheet of ice is in the form of a circle and maintains this shape as it melts. If the ice is melting in such a way that the area of the sheet is decreasing at a rate of 0.5 m^2/sec, at what rate is the radius decreasing when the area of the sheet is 12 m^2?

Solution: we can call the area A and radius r, so $A = \pi r^2$

We also know that $\frac{dA}{dt} = -0.5$.

We want to know $\frac{dr}{dt}$ when $A = 12$

Thus, we differentiate with respect to t to get

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

and can find the radius using the original area formula and the known point $A = 12$

$12 = \pi r^2$ gives us $r = \sqrt{(12/\pi)}$

and the rate of change at this point is therefore

$$-0.5 = 2\pi \sqrt{(12/\pi)} \frac{dr}{dt}$$

which can be solved for $\frac{dr}{dt}$

$$\frac{dr}{dt} = -\frac{1}{4\pi\sqrt{(12/\pi)}} = -\frac{1}{8\sqrt{3\pi}}$$

Therefore the radius is *decreasing* at a rate of $\boxed{\frac{1}{8\sqrt{3\pi}} \text{ m/sec}}$