1. (20 pts) The following problems are unrelated. A list of helpful trigonometric identities is provided on the back of the exam.

(a) Suppose that $\sec(\theta) = 3$, where $\frac{3\pi}{2} \leq \theta \leq 2\pi$. Find the values of the following

(i) $\sin(\theta)$ (ii) $\cos(\theta)$ (iii) $\cos(2\theta)$

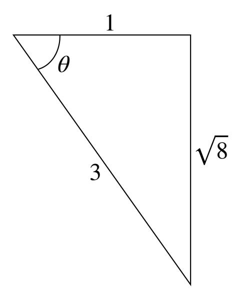
(b) Find all values of x in the interval $[0, \pi]$ that solve $2\cos^2(x) = 2 - \cos(2x)$.

Solution:

(a) Since $\sec(\theta) = 3$ and θ is in quadrant IV, we can draw the acute right triangle with angle θ with adjacent side of length 1 and hypotenuse of length 3. The remaining side length, opp, can be found via the Pythagorean theorem

$$1^2 + \text{opp}^2 = 3^2 \implies \text{opp} = \sqrt{8}$$

The acute right triangle can then resembles



The remaining trig functions can be computed

(i) $\sin\theta = -\frac{\sqrt{8}}{3}$ (SOH and fact that θ is in quadrant 4)

(ii) $\cos\theta = \frac{1}{3}$ (CAH)

(iii) $\cos 2\theta = \cos^2\theta - \sin^2\theta = \left(\frac{1}{3}\right)^2 - \left(\frac{\sqrt{8}}{3}\right)^2 = -\frac{7}{9}$

(b) Using the identity $\cos^2(x) = \frac{1+\cos(2x)}{2}$ we can rewrite the equation as

$$1 + \cos(2x) = 2 - \cos(2x)$$
$$\implies 2\cos(2x) - 1 = 0$$
$$\implies \cos(2x) = \frac{1}{2}$$

Notice that $\cos(\theta) = 1/2$ when $\theta = \pi/3$ and $\theta = 5\pi/3$. Here $2x = \theta$, so we find that the appropriate values of x in $[0, \pi]$ are

$$x = \frac{\pi}{6}, \quad \frac{5\pi}{6}$$

2. (20 pts) Evaluate the following limits and simplify your answers.

(a) $\displaystyle\lim_{x\to-3} \frac{\sqrt{2x+22}-4}{x+3}$

(b) $\displaystyle\lim_{x\to\infty} \frac{20x^4-7x^3}{2x+9x^2+5x^4}$

(c) $\displaystyle\lim_{x\to0} x^4 \sin\left(\frac{\pi}{x}\right)$

Solution:

(a) We compute this limit by multiplying by the conjugate

$$\begin{aligned}\lim_{x\to-3} \frac{\sqrt{2x+22}-4}{x+3}\frac{\sqrt{2x+22}+4}{\sqrt{2x+22}+4} &= \lim_{x\to-3} \frac{2x+22-16}{(x+3)(\sqrt{2x+22}+4)} \\ &= \lim_{x\to-3} \frac{2(x+3)}{(x+3)(\sqrt{2x+22}+4)} \\ &= \lim_{x\to-3} \frac{2}{\sqrt{2x+22}+4} = \frac{2}{8} = \boxed{\frac{1}{4}}\end{aligned}$$

(b) We can factor out x^4 in the numerator and denominator (this is equivalent to multiplying both the numerator and denominator by $1/x^4$)

$$\begin{aligned}\lim_{x\to\infty} \frac{x^4(20-\frac{7}{x^3})}{x^4(\frac{2}{x^3}+\frac{9}{x^2}+5)} &= \lim_{x\to\infty} \frac{20-7/x^3}{2/x^3+9/x^2+5} \\ &= \frac{20}{5} = 4\end{aligned}$$

(c) Here we need to apply the squeeze theorem. We note that

$$-1 \leq \sin(\frac{\pi}{x}) \leq 1$$

for all values of x in the domain of the trig function. From this we can conclude

$$-x^4 \leq x^4 \sin(\frac{\pi}{x}) \leq x^4$$

and indentify $f(x) = -x^4$, $h(x) = x^4$. We can now take the limits of $f(x)$ and $h(x)$ as

$$\lim_{x\to0} x^4 = \lim_{x\to0} -x^4 = 0$$

Therefore, by the Squeeze theorem, we may conclude

$$\lim_{x\to0} x^4 \sin(\frac{\pi}{x}) = 0.$$

3. (20 pts) In this problem, let $f(x) = \sqrt{x-6}$, $g(x) = |2x-2|$.

(a) Sketch the function $g(x)$ on the interval $[-2, 2]$. Label all intercepts.

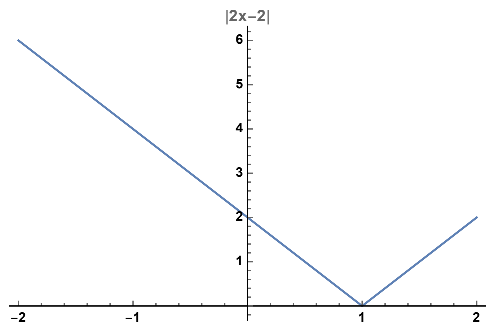
(b) Find $(f \circ g)(x)$.

(c) Evaluate the limit: $\lim_{x\to 1} \frac{g(x)}{2-2x}$.

(d) Suppose we let $h(x) = \begin{cases} f(x) & \text{if } x > 6 \\ g(x) & \text{if } x \leq 6 \end{cases}$. Are there any values of x for which $h(x)$ is not continuous? Justify your answer and specify the type of discontinuity (i.e. jump, removable, infinite) if there are any.

Solution:

(a) The plot of $g(x) = |2x - 2$ is the following



(b)

$$f \circ g(x) = \sqrt{|2x-2| - 6}$$

(c) It is helpful to note that $g(x) = \begin{cases} 2x-2 & x \geq 2 \\ -2x+2 & x < 2 \end{cases}$. we evaluate the limit by computing one-sided limits

$$\begin{aligned} \lim_{x\to 1^-} \frac{g(x)}{2-2x} &= \lim_{x\to 1^-} \frac{-2x+2}{2-2x} \\ &= 1 \end{aligned}$$

$$\begin{aligned} \lim_{x\to 1^+} \frac{g(x)}{2-2x} &= \lim_{x\to 1^+} \frac{2x-2}{2-2x} \\ &= -1 \end{aligned}$$

Since the left and right hand limits are not equal, then

$$\boxed{\lim_{x\to 1} \frac{g(x)}{2-2x} \quad DNE}$$

(d) Since $f(x)$ is continuous for all $x > 6$ and $g(x)$ is continuous for $x \leq 6$, the only point we need to check for a discontinuity is $x = 6$. Let's compute left and right-hand limits

$$\lim_{x\to 6^-} h(x) = \lim_{x\to 6^-} |2x-2| = 10.$$

$$\lim_{x\to 6^+} h(x) = \lim_{x\to 6^+} \sqrt{x-6} = 0$$

So $h(x)$ is discontinuous at $x = 6$. The discontinuity is a $\boxed{\text{jump}}$ discontinuity.

4. (20 pts) In this problem, we will study the function $f(x) = \dfrac{\cos x \sec x \sin x}{3x}$

(a) Compute $f(\pi)$ and $f(\frac{\pi}{4})$.

(b) Classify $f(x)$ as even, odd, or neither. For full credit you must show your work.

(c) What is $\lim\limits_{x \to 0} f(x)$?

(d) Is $f(x)$ continuous at $x = 0$? If the function is **not** continuous, classify the type of discontinuity.

Solution: It is helpful (but not necessary) to rewrite $f(x) = \dfrac{\cos(x)\sin(x)}{3x\cos(x)}$

(a)

$$\begin{aligned} f(\pi) &= \frac{\cos\pi \sin\pi}{3\pi\cos(\pi)} \\ &= \frac{1 \cdot 0}{3\pi \cdot 1} \\ &= 0 \end{aligned}$$

$$\begin{aligned} f(\pi/4) &= \frac{\cos(\pi/4)\sin(\pi/4)}{3(\pi/4)\cos(\pi/4)} \\ &= \frac{(\sqrt{2}/2)}{(3\pi/4)} \\ &= \frac{\sqrt{2}}{2}\frac{4}{3\pi} \\ &= \frac{2\sqrt{2}}{3\pi} \end{aligned}$$

(b) We check if the function is even/odd by computing

$$\begin{aligned} f(-x) &= \frac{\cos(-x)\sec(-x)\sin(-x)}{3(-x)} \\ &= \frac{\cos(x)\sec(x)\,[-\sin(x)]}{-3x} \\ &= \frac{\cos(x)\sec(x)\sin(x)}{3x} \\ &= f(x) \end{aligned}$$

Since $f(-x) = f(x)$, the function is $\boxed{\text{even}}$.

(c)

$$\begin{aligned} \lim_{x\to 0} f(x) &= \lim_{x\to 0} \frac{\cos x \sin x}{3x\cos(x)} \\ &= \lim_{x\to 0}\frac{\sin x}{3x} = 1/3 \cdot \lim_{x \to 0}\frac{\sin x}{x} = 1/3 \cdot 1 = \boxed{1/3} \end{aligned}$$

(d) $f(x)$ is not continuous at at $x = 0$ because 0 is not in the domain of the function. Since the limit exists as $x \to 0$, this indicates there is a $\boxed{\text{removable discontinuity}}$

5. (20 pts) The following problems are unrelated.

(a) Identify all vertical and horizontal asymptotes of the function $f(x) = \dfrac{x+7}{x^2-4}$. Show all your work.

(b) What is the domain of the function $g(x) = \dfrac{1}{\sqrt{x^2(x-1)}}$? Express your answer using interval notation.

(c) Use the Intermediate Value Theorem to show that there is a solution to $x = x^3 + 5$.

Solution:

(a) There are possible VAs at $x = \pm 2$, to verify we can compute the following one-sided limits (note: only a single one-sided limit is required at $x = \pm 2$ to establish the presence of a VA)

$$\lim_{x\to -2^+} \frac{x+7}{(x-2)(x+2)} = \frac{5}{-4(0^+)} = -\infty$$
$$\lim_{x\to -2^-} \frac{x+7}{(x-2)(x+2)} = \frac{5}{-4(0^-)} = +\infty$$
$$\lim_{x\to 2^+} \frac{x+7}{(x-2)(x+2)} = \frac{9}{(0^+)(4)} = +\infty$$
$$\lim_{x\to 2^-} \frac{x+7}{(x-2)(x+2)} = \frac{9}{(0^-)(4)} = -\infty$$

Therefore there are VAs at $x = \pm 2$.

We compute HAs via the limits as $x \to \pm\infty$

$$\lim_{x\to \infty} \frac{x+7}{(x-2)(x+2)} = 0$$
$$\lim_{x\to -\infty} \frac{x+7}{(x-2)(x+2)} = 0$$

so $y = 0$ is the only horizontal asymptote.

(b) The function is defined whenever the denominator is nonzero, or the function inside the square root is positive. This amounts to

$$x^2(x-1) > 0$$

Which is valid provided x is in the domain $\boxed{(1, \infty)}$.

(c) The equation can be rewritten as

$$x^3 - x + 5 = 0.$$

Now we need to show that the function $f(x) = x^3 - x + 5$ has a root, i.e. a value $x = c$ such that $x = 0$.

First we need to find two x values such that $f(x)$ is negative and positive respectively. We can, for example, pick $x = -2$ and $x = -1$ so that

$$f(-2) = (-2)^3 - (-2) + 5 = -1,$$
$$f(-1) = (-1)^3 - (-1) + 5 = 5$$

Now, since $f(x)$ is continuous and $f(-2) < 0$ and $f(-1) > 0$, then we can use the **intermediate value theorem** to conclude there is some value $x = c$ such that $f(c) = 0$. At this point,

$$c^3 - c + 5 = 0 \implies c^3 + 5 = c$$