1. (34pts) The following problems are not related.

(a)(17pts) Use *logarithmic differentiation* to find $\dfrac{dy}{dx}$ if $y = \dfrac{x^x}{\cosh(x)}$.

(b)(17pts)(i)(8pts) Write down the piecewise definition of the function $f(x) = |e^x - 2|$.

(ii)(9pts) Evaluate the integral $\displaystyle\int_0^{\ln(4)} |e^x - 2|\,dx$. Simplify your answer.

Solution: (a)(17pts) Note that

$$\ln(y) = \ln\left(\frac{x^x}{\cosh(x)}\right) = x\ln(x) - \ln(\cosh(x)) \Rightarrow \frac{y'}{y} = \ln(x) + x\cdot\frac{1}{x} - \frac{1}{\cosh(x)}\cdot\sinh(x) = \ln(x) + 1 - \tanh(x)$$

thus

$$\frac{y'}{y} = \ln(x) + 1 - \tanh(x) \Rightarrow y' = y\big[\ln(x) + 1 - \tanh(x)\big] \Rightarrow \boxed{y' = \frac{x^x}{\cosh(x)}\big[\ln(x) + 1 - \tanh(x)\big]}$$

(b)(i)(8pts) Note that $e^x - 2 = 0 \Rightarrow e^x = 2 \Rightarrow x = \ln(2)$ thus

$$|e^x - 2| = \begin{cases} e^x - 2, & \text{if } x \geq \ln(2) \\ -(e^x - 2), & \text{if } x < \ln(2) \end{cases} \Rightarrow \boxed{|e^x - 2| = \begin{cases} e^x - 2, & \text{if } x \geq \ln(2) \\ 2 - e^x, & \text{if } x < \ln(2) \end{cases}}$$

(b)(ii)(9pts) Now note that

$$\begin{aligned} \int_0^{\ln(4)} |e^x - 2|\,dx &= \int_0^{\ln(2)} (2 - e^x)\,dx + \int_{\ln(2)}^{\ln(4)} (e^x - 2)\,dx \\ &= (2x - e^x)\Big|_0^{\ln(2)} + (e^x - 2x)\Big|_{\ln(2)}^{\ln(4)} \\ &= (2\ln(2) - e^{\ln(2)}) - (2\cdot 0 - e^0) + (e^{\ln(4)} - 2\ln(4)) - (e^{\ln(2)} - 2\ln(2)) \\ &= 2\ln(2) - 2 + 1 + 4 - 2\ln(4) - 2 + 2\ln(2) \\ &= \ln(4) + 1 - 2\ln(4) + \ln(4) = \boxed{1} \end{aligned}$$

---

2. (35pts) The following problems are not related.

(a)(16pts) In this problem, you will find two positive integers x and y such that the sum of the first number and four times the second number is 1000 and the product is as large as possible. Answer the following questions: (i)(4pts) Is this is a *minimization* or *maximization* problem? Write down a function in terms of the two variables x and y that you would minimize (or maximize). (ii)(4pts) Write down an equation that relates the variables x and y. (iii)(8pts) Now using optimization find the value of x and y that satisfy this problem. Justify your answer by classifying your critical point(s) using either the 1st or 2nd Derivative Test.

(b)(16pts) Evaluate the integral, show all work: $\displaystyle\int_e^{e^3} \frac{1}{x\ln(x)}\,dx$.

(c)(3pts) **Fill in the blank, no justification necessary.** If $f(x) = x^4 + x + 3$ for $0 \leq x \leq 2$ then the point c in $(0, 2)$ whose existence is guaranteed by the Mean Value Theorem is $c =$ ___*write answer in bluebook*___.

Solution:

(a)(i)(4pts) This is a **maximization** problem and we wish to $\boxed{\text{maximize } P = xy.}$

(a)(*ii*)(4pts) The constraint is $\boxed{x + 4y = 1000}$ which implies either $y = 250 - \dfrac{x}{4}$ or $x = 1000 - 4y$.

(a)(*iii*)(8pts) We have

$$P = xy = x\left(250 - \frac{x}{4}\right) = 250x - \frac{x^2}{4} \Rightarrow \frac{dP}{dx} = 250 - \frac{x}{2} \text{ and } \frac{dP}{dx} = 0 \Rightarrow \boxed{x = 500 \text{ and } y = 125}$$

or

$$P = xy = (1000 - 4y)y = 1000y - 4y^2 \Rightarrow \frac{dP}{dy} = 1000 - 8y \text{ and } \frac{dP}{dy} = 0 \Rightarrow y = \frac{1000}{8} = 125 \text{ and } x = 500.$$

Finally note that, for example, $\dfrac{d^2P}{dx} = -\dfrac{1}{2} < 0$ (or $\dfrac{d^2P}{dy} = -8 < 0$) which implies, by the <u>2nd Derivative Test</u> that $x = 500$ is a (local and absolute) maximum.

(b)(16pts) Here if we let $u = \ln(x)$ then $du = dx/x$ and furthermore if $x = e$ then $u = \ln(e) = 1$ and if $x = e^3$ then $u = \ln(e^3) = 3$ thus

$$\int_e^{e^3} \frac{dx}{x\ln(x)} = \int_1^3 \frac{du}{u} = \ln(u)\Big|_1^3 = \ln(3) - \ln(1) = \boxed{\ln(3).}$$

(c)(3pts) $\boxed{\text{Note that } c = 2^{1/3}.}$ We have $f'(x) = 4x^3 + 1$ and, by the Mean Value Theorem, there exists a number $0 < c < 2$ such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} \Rightarrow 4c^3 + 1 = \frac{21 - 3}{2} = 9 \Rightarrow 4c^3 = 8 \Rightarrow \boxed{c = \sqrt[3]{2}}$$

---

3. (34pts) The following problems are not related.

(a)(17pts) Evaluate the limit: $\lim\limits_{x\to 0^+} x^{\sqrt{x}}$. Show all work.

(b)(17pts) If $f(x) = \displaystyle\int_2^x \frac{1}{\sqrt{1+t^4}}\,dt$, find $(f^{-1})'(0)$. Show all work.

Solution: (a)(17pts) Note that if we try to evaluate the limit we get the indeterminate form "0^0" thus if we let $y = \lim\limits_{x\to 0^+} x^{\sqrt{x}}$ then, applying the natural log to both sides and using the properties of continuity and the natural log yields

$$\ln(y) = \ln\big(\lim_{x\to 0^+} x^{\sqrt{x}}\big) = \lim_{x\to 0^+} \ln\left(x^{\sqrt{x}}\right) = \lim_{x\to 0^+} \sqrt{x}\ln(x)$$

and $\lim\limits_{x\to 0^+} \sqrt{x}\cdot\ln(x)$ yields the indeterminate form "$0 \cdot -\infty$" thus

$$\lim_{x\to 0^+} \sqrt{x}\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{1/\sqrt{x}} \overset{L'H}{=} \lim_{x\to 0^+} \frac{(1/x)}{-1/2x^{3/2}} = \lim_{x\to 0^+} -\frac{2x^{3/2}}{x} = \lim_{x\to 0^+} -2x^{1/2} = 0 \Rightarrow \boxed{\lim_{x\to 0^+} x^{\sqrt{x}} = e^0 = 1}$$

(b)(17pts) First note that $f(2) = 0 \Rightarrow 2 = f^{-1}(0)$ and $f'(x) = \dfrac{1}{\sqrt{1+x^4}}$ thus, by a theorem from Ch. 5, we have

$$(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'(2)} = \frac{1}{1/\sqrt{1+(2)^4}} = \boxed{\sqrt{17}.}$$

---

4. (35pts) The following problems are not related.

(a)(16pts) Is the function $f(x) = \begin{cases} \dfrac{\sin(-x)}{6x}, & \text{if } x < 0 \\ \dfrac{x^2+x+2}{6x^2-12}, & \text{if } x \geq 0 \end{cases}$ *continuous* at $x = 0$? Justify your answer with <u>limits</u>.

(b)(16pts) Evaluate $\displaystyle\int \frac{1+2x}{1+x^2}\,dx$. (*Hint:* Recall that $\int[f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$.)

(c)(3pts) Which graph below best matches the graph of the function $f(x) = 8x^3 - 2x^4$? **Choose only <u>one</u> answer.** *No justification necessary, clearly indicate your answer otherwise points will be deducted.*

Solution: (a)(16pts) Note that

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin(-x)}{6x} \overset{L'H}{=} \lim_{x\to 0^-} \frac{\cos(-x)\cdot(-1)}{6} = -\frac{1}{6}$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x^2+x+2}{6x^2-12} = \frac{0+2}{0-12} = -\frac{1}{6} = f(0)$$

thus we have shown

$$\boxed{\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) \text{ thus } f(x) \text{ is continuous at } x = 0.}$$

(b)(16pts) Using the hint we have

$$\int \frac{1+2x}{1+x^2}\,dx = \int \frac{1}{1+x^2}\,dx + \int \frac{2x}{1+x^2}\,dx$$

now note that $\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$ and if we use the u-substitution $u = 1 + x^2$ we get $du = 2xdx$ thus

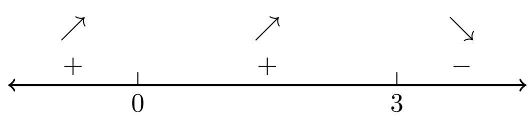
$$\int \frac{2x}{1+x^2}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|1+x^2| + C = \ln(1+x^2) + C$$

thus

$$\int \frac{1+2x}{1+x^2}\,dx = \int \frac{1}{1+x^2}\,dx + \int \frac{2x}{1+x^2}\,dx = \boxed{\tan^{-1}(x) + \ln(1+x^2) + C.}$$

(c)(3pts) $\boxed{\text{Graph B.}}$ Note that $f(x) = 8x^3 - 2x^4 = 2x^3(4-x)$ thus the curve has x-intercepts at $(0,0)$ and $(4,0)$, thus we can eliminate Graph D. Also note that $f'(x) = 24x^2 - 8x^3 = 8x^2(3-x)$ thus $f'(x) = 0 \Rightarrow x = 0, 3$ and note that

1st Deriv. Test for $f(x)$



Sign chart for $f'(x) = 8x^2(3-x)$

thus we can eliminate Graph C. Finally, note that there is a local maximum at $x = 3$ with local maximum value of $f(3) = 2\cdot 3^3 \cdot (4-3) = 54$ thus we can eliminate Graph A, leaving only Graph B.

---

5. (12pts) Answer either **ALWAYS TRUE** or **FALSE**. You do <u>NOT</u> need to justify your answer. (*Don't just write down "A.T." or "F", completely write out the words "ALWAYS TRUE" or "FALSE" depending on your answer.*)

(a)(3pts) The linearization of the function $f(x) = \frac{e^{2x}}{x}$ at $a = 1$ is $y = e^2 x$.

(b)(3pts) It can be shown that $\lim_{h\to 0} \frac{\sin^{-1}(2x+2h)}{h} - \frac{\sin^{-1}(2x)}{h} = \frac{2}{\sqrt{1-4x^2}}$.

(c)(3pts) The function $f(x) = xe^x$ has a local minimum at $x = -1$.

(d)(3pts) If $f(x)$ is an even function and $\int_0^2 f(x)dx = \pi - 3$ then $\int_{-2}^0 f(x)dx = -\pi + 3$.

Solution: $\boxed{\text{3pts each: (a) ALWAYS TRUE (b) ALWAYS TRUE (c) ALWAYS TRUE (d) FALSE}}$

<u>Discussion:</u>

(a) Note that

$$f'(x) = \frac{d}{dx}\left[\frac{e^{2x}}{x}\right] = \frac{2e^{2x}\cdot x - e^{2x}\cdot 1}{x^2} = \frac{e^{2x}(2x-1)}{x^2} \Rightarrow f'(1) = e^2$$

thus

$$L(x) = f(1) + f'(1)(x-1) = e^2 + e^2(x-1) = e^2 x$$

(b) Note that if we let $f(x) = \sin^{-1}(2x)$ then

$$\lim_{h\to 0} \frac{\sin^{-1}(2x+2h) - \sin^{-1}(2x)}{h} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \frac{d}{dx} f(x) = \frac{d}{dx}\left[\sin^{-1}(2x)\right] = \frac{1}{\sqrt{1-(2x)^2}} \cdot 2 = \frac{2}{\sqrt{1-4x^2}}$$

(c) Note that

$$f'(x) = e^x + xe^x = e^x(1+x) \text{ and } f'(x) = 0 \Rightarrow x = -1$$

and also note that

$$f''(x) = e^x(1+x) + e^x \Rightarrow f''(-1) = e^{-1} > 0 \Rightarrow x = -1 \text{ is a local minimum.}$$

(d) If $f(x)$ is an even function then $f(-x) = f(x)$ and so

$$\pi - 3 = \int_0^2 f(x)dx = \int_0^2 f(-x)dx$$

and if we let $u = -x$ then $du = -dx \Rightarrow -du = dx$ and if $x = 0$ then $u = 0$ and if $x = 2$ then $u = -2$ thus

$$\pi - 3 = \int_0^2 f(x)dx = \int_0^2 f(-x)dx = \int_0^{-2} f(u) \cdot -du = -\int_0^{-2} f(u)\,du = \int_{-2}^0 f(u)\,du = \int_{-2}^0 f(x)\,dx$$

where the last equality follows from the fact that the "dummy variable" u is replaced with x, thus we see that

$$\int_{-2}^0 f(x)\,dx = \pi - 3 \neq -\pi + 3.$$
