1. (24pts) The following problems are not related.

   (a)(12pts) Find the value of the number a so that the function $f(x) = \dfrac{ax+5}{x^2-1}$ has a critical point at $x = 2$. Show all work.

   (b)(12pts) Suppose $g'(x) = 4x^3(x-1)^3 + 3x^4(x-1)^2$. Find the local maximum and local minimum of the function $g(x)$ and justify your answer with either the 1st or 2nd Derivative Test. Clearly label your answers.

Solution: (a)(12pts) Differentiating yields

$$f'(x) = \frac{d}{dx}\left[\frac{ax+5}{x^2-1}\right] = \frac{a(x^2-1)-(ax+5)\cdot 2x}{(x^2-1)^2} = \frac{ax^2-a-2ax^2-10x}{(x^2-1)^2} = \frac{-a-ax^2-10x}{(x^2-1)^2}$$

and since $f(x)$ has a critical point at $x = 2$ this implies that $f'(2) = 0$ thus

$$f'(2) = \left.\frac{-a-ax^2-10x}{(x^2-1)^2}\right|_{x=2} = \frac{-a-4a-20}{(4-1)^2} = \frac{-5a-20}{9} = 0 \Rightarrow -5a-20 = 0 \Rightarrow \boxed{a = -\frac{20}{5} = -4}$$

so $f(x) = \dfrac{-4x+5}{x^2-1}$.

(b)(12pts) Factoring out the common terms in $g'(x)$ yields

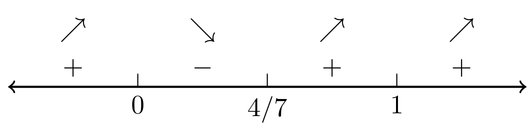
$$4x^3(x-1)^3 + 3x^4(x-1)^2 = x^3(x-1)^2\Big[4(x-1)+3x\Big] = x^3(x-1)^2[4x-4+3x] = x^3(x-1)^2(7x-4)$$

and, to find the critical points, we set $g'(x) = 0$:

$$g'(x) = 0 \Rightarrow x^3(x-1)^2(7x-4) = 0 \Rightarrow x = 0, 4/7, 1 \text{ are critical points of } g(x).$$

Now to classify, using the First Derivative test yields

1st Derivative Test



Sign chart for $f'(x) = x^3(x-1)^2(7x-4)$

so we have a $\boxed{\text{local maximum at } x = 0 \text{ and a local minimum at } x = 4/7.}$

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2. (28pts) The following problems are not related.

   (a)(12pts) Set-up, **but do not evaluate**, a Riemann sum to estimate the area under the curve $f(x) = x\sin(x)$ from $x = 0$ to $x = \pi$ using five rectangles of equal width and left endpoints. (*NOTE:* Do not leave your answer in terms of $f(x)$, you should set up your equation to the point that the only thing left to do is evaluate the equation but then **do not evaluate it**.)

   (b)(12pts) Water flows from the bottom of a storage tank at a rate of $r(t) = 200 - 4t$ liters per minute, where $0 \le t \le 50$. Find the amount of water that flows from the tank during the first 10 minutes. Simplify your answer. Show all work.

   (c)(4pts) Which limit below is equal to $\int_0^{\pi} x\sin(x)\,dx$? **Choose only <u>one</u> answer.** *No justification necessary, copy down the entire answer. If you do not copy down the entire answer, points will be deducted.*

(A) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi i}{n} \sin\left(\frac{\pi i}{n}\right)$ (B) $\lim_{i\to\infty} \sum_{i=1}^{n} \frac{\pi i}{n} \sin\left(\frac{\pi i}{n}\right) \frac{\pi}{n}$ (C) $\lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(\frac{\pi i}{n}\right) \frac{\pi}{n}$ (D) $\lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(\frac{\pi i}{n}\right) \frac{\pi^2 i}{n^2}$

Solution:

(a)(12pts) Note that $\Delta x = \dfrac{b-a}{n} = \dfrac{\pi - 0}{5} = \pi/5$ and $x_i = a + \Delta x = 0 + \dfrac{\pi i}{5} = \dfrac{\pi i}{5}$ for $i = 0, 1, \ldots, 5$ and, since left endpoints have been specified, we have $x_i^* = x_{i-1}$ thus

$$\begin{aligned}
\int_0^{\pi} x\sin(x)\,dx \approx L_5 &= \sum_{i=1}^{5} f(x_{i-1})\Delta x \\
&= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\
&= \boxed{0\cdot\sin(0)\cdot\frac{\pi}{5} + \frac{\pi}{5}\sin\left(\frac{\pi}{5}\right)\cdot\frac{\pi}{5} + \frac{2\pi}{5}\sin\left(\frac{2\pi}{5}\right)\cdot\frac{\pi}{5} + \frac{3\pi}{5}\sin\left(\frac{3\pi}{5}\right)\cdot\frac{\pi}{5} + \frac{4\pi}{5}\sin\left(\frac{4\pi}{5}\right)\cdot\frac{\pi}{5}} \\
&= \frac{\pi^2}{25}\left[\sin\left(\frac{\pi}{5}\right) + 2\sin\left(\frac{\pi}{5}\right) + 3\sin\left(\frac{\pi}{5}\right) + 4\sin\left(\frac{\pi}{5}\right)\right]
\end{aligned}$$

(b)(12pts) Here we have

$$\int_0^{10} r(t)\,dt = \int_0^{10} (200 - 4t)\,dt = \left(200t - \frac{4t^2}{2}\right)\Bigg|_0^{10} = \left(200\cdot 10 - 2\cdot 10^2\right) - \left(200\cdot 0 - 2\cdot 0^2\right) = 2000 - 200 = \boxed{1800 \text{ liters.}}$$

(c)(4pts) $\boxed{\text{Choice D.}}$ Note that $\Delta x = \dfrac{b-a}{n} = \pi/n$ and $x_i = a + i\Delta x = \dfrac{\pi i}{n}$ and if we use right endpoints then $x_i^* = x_i$ thus

$$\int_0^{\pi} x\sin(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} \frac{\pi i}{n}\sin\left(\frac{\pi i}{n}\right)\frac{\pi}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}\sin\left(\frac{\pi i}{n}\right)\frac{\pi^2 i}{n^2} \Rightarrow \text{Choice D.}$$

Note that in Choice B, the limit is evaluated as $i \to \infty$ which is incorrect.

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3. (20pts) The following problems are not related.

(a)(10pts) Suppose we want to approximate the value of $\sqrt[3]{2}$ using Newton's Method. What would the formula for x_{n+1} be? (To get full points for this question you must provide the explicit formula for x_{n+1} in terms of x_n, the generic formula for Newton's Method is <u>not</u> sufficient. You do **not** need to approximate the solution.)

(b)(10pts) Find the most general antiderivative of the function $f(x) = \sqrt[3]{x} + \sec^2(x) + \pi^2$. Show all work.

Solution: (a)(10pts) Let $x = \sqrt[3]{2} = 2^{1/3}$ then $x^3 = 2 \Rightarrow x^3 - 2 = 0$. Thus, we wish to approximate the root of $f(x) = x^3 - 2$ using Newton's Method. Note that $f'(x) = 3x^2$ thus

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 2}{3x_n^2}} = \frac{3x_n^3 - (x_n^3 - 2)}{3x_n^2} = \frac{3x_n^3 - x_n^3 + 2}{3x_n^2} = \frac{2x_n^3 + 2}{3x_n^2} \quad \text{for } n = 1, 2, \ldots$$

(b)(10pts) Note that

$$\int (x^{1/3} + \sec^2(x) + \pi^2)\,dx = \frac{x^{1/3+1}}{1/3+1} + \tan(x) + \pi^2 x + C = \boxed{\frac{3}{4}x^{4/3} + \tan(x) + \pi^2 x + C}$$

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4. (28pts) The following problems are not related.

(a)(12pts) If $\displaystyle\int_0^1 x\sqrt{x^2+4}\,dx = 5\sqrt{5} - 8$, find $\displaystyle\int_1^0 \left[t\sqrt{t^2+4} - 10\right] dt$. Show all work.

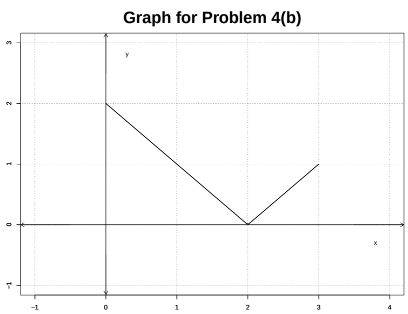
(b)(12pts) Evaluate the definite integral $\displaystyle\int_0^3 |x-2|\,dx$. Show all work. Simplify your answer.

(c)(4pts) Suppose, for the function $f(x)$, we have $f'(x) = \displaystyle\int \frac{6(3x^2+1)}{(x^2-1)^3}\,dx$. Which graph below best matches the graph of the function $f(x)$? (*Hint:* Do not try to solve the integral, you already have all the information needed to solve this problem.) **Choose only <u>one</u> answer.** *No justification necessary, clearly indicate your answer otherwise points will be deducted.*

Solution: (a)(12pts) Note that, by changing the "dummy variable", we have $\int_0^1 t\sqrt{t^2+4}\,dt = \int_0^1 x\sqrt{x^2+4}\,dx = 5\sqrt{5}-8$ thus

$$\begin{aligned}\int_1^0 \left[t\sqrt{t^2+4}-10\right] dt &= -\int_0^1 \left[t\sqrt{t^2+4}-10\right] dt \\ &= -\int_0^1 \left[t\sqrt{t^2+4}\right] dt + \int_0^1 10\,dt = -\left(5\sqrt{5}-8\right) + 10t\Big|_0^1 = -5\sqrt{5}+8+10 = \boxed{18-5\sqrt{5}.}\end{aligned}$$

(b)(12pts) This problem can be done geometrically or directly. Note that the graph of $f(x) = |x-2|$ looks like: and



so we see the region of interest can be described by two triangles, one triangle with base and height length of 2 units and the other triangle with base and height length of 1 unit, thus

$$\int_0^3 |x-2|\,dx = \frac{1}{2}\cdot 2\cdot 2 + \frac{1}{2}\cdot 1\cdot 1 = \frac{4}{2}+\frac{1}{2} = \frac{5}{2}$$

We can also do the integral directly by applying the definition of the absolute value and separating the integral:

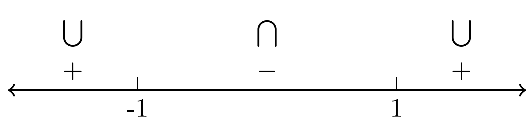
$$\int_0^3 |x-2|\,dx = \int_0^2 |x-2|\,dx + \int_2^3 |x-2|\,dx = \int_0^2 -(x-2)\,dx + \int_2^3 (x-2)\,dx$$

thus,

$$\begin{aligned}\int_0^3 |x-2|\,dx &= \int_0^2 -(x-2)\,dx + \int_2^3 (x-2)\,dx \\ &= -\left(\frac{x^2}{2}-2x\right)\Big|_0^2 + \left(\frac{x^2}{2}-2x\right)\Big|_2^3 \\ &= -\left(\frac{2^2}{2}-2\cdot 2\right) + 0 + \left(\frac{3^2}{2}-2\cdot 3\right) - \left(\frac{2^2}{2}-2\cdot 2\right) = 2+\frac{9}{2}-6+2 = \frac{9}{2}-2 = \boxed{\frac{5}{2}}\end{aligned}$$

(c)(4pts) $\boxed{\text{Choice C.}}$ Note that $f'(x) = \int \frac{6(3x^2+1)}{(x^2-1)^3}\,dx$ implies that $f'(x)$ is an antiderivative of $\frac{6(3x^2+1)}{(x^2-1)^3}$ and so $f''(x) = \frac{6(3x^2+1)}{(x^2-1)^3}$ thus we can determine the concavity of the curve $f(x)$. Note

Concavity of $f(x)$



Sign chart for $f''(x) = \frac{6(3x^2+1)}{(x^2-1)^3}$

and choice C best fits this concavity pattern.