1. (24pts) The following problems are not related.
   
   (a)(12pts) Find the value of the number \( a \) so that the function \( f(x) = \frac{ax + 5}{x^2 - 1} \) has a critical point at \( x = 2 \). Show all work.

   Solution: Differentiating yields
   
   \[
   f'(x) = \frac{d}{dx} \left[ \frac{ax + 5}{x^2 - 1} \right] = \frac{a(x^2 - 1) - (ax + 5) \cdot 2x}{(x^2 - 1)^2} = \frac{ax^2 - a - 2ax^2 - 10x}{(x^2 - 1)^2} = \frac{-a - ax^2 - 10x}{(x^2 - 1)^2}
   \]
   
   and since \( f(x) \) has a critical point at \( x = 2 \) this implies that \( f'(2) = 0 \) thus
   
   \[
   f'(2) = \frac{-a - ax^2 - 10x}{(x^2 - 1)^2} \bigg|_{x=2} = \frac{-a - 4a - 20}{(4 - 1)^2} = \frac{-5a - 20}{9} = 0 \Rightarrow -5a - 20 = 0 \Rightarrow a = \frac{-20}{5} = -4
   \]
   
   so \( f(x) = \frac{-4x + 5}{x^2 - 1} \).

   (b)(12pts) Suppose \( g'(x) = 4x^3(x - 1)^3 + 3x^4(x - 1)^2 \). Find the local maximum and local minimum of the function \( g(x) \) and justify your answer with either the 1st or 2nd Derivative Test. Clearly label your answers.

   Solution: Factoring out the common terms in \( g'(x) \) yields
   
   \[
   4x^3(x - 1)^3 + 3x^4(x - 1)^2 = x^3(x - 1)^2[4(x - 1) + 3x] = x^3(x - 1)^2[4x - 4 + 3x] = x^3(x - 1)^2(7x - 4)
   \]
   
   and, to find the critical points, we set \( g'(x) = 0 \):
   
   \[
   g'(x) = 0 \Rightarrow x^3(x - 1)^2(7x - 4) = 0 \Rightarrow x = 0, 4/7, 1 \text{ are critical points of } g(x).
   \]
   
   Now to classify, using the First Derivative test yields
   
   1st Derivative Test
   
   \[
   \begin{array}{c|c|c|c|c}
   & + & - & + & + \\
   \hline
   0 & 4/7 & 1 &  &  \\
   \end{array}
   \]
   
   Sign chart for \( f'(x) = x^3(x - 1)^2(7x - 4) \)
   
   so we have a local maximum at \( x = 0 \) and a local minimum at \( x = 4/7 \).

2. (28pts) The following problems are not related.

   (a)(12pts) Set-up, but do not evaluate, a Riemann sum to estimate the area under the curve \( f(x) = x \sin(x) \) from \( x = 0 \) to \( x = \pi \) using five rectangles of equal width and left endpoints. (NOTE: Do not leave your answer in terms of \( f(x) \), you should set up your equation to the point that the only thing left to do is evaluate the equation but then do not evaluate it.)

   (b)(12pts) Water flows from the bottom of a storage tank at a rate of \( r(t) = 200 - 4t \) liters per minute, where \( 0 \leq t \leq 50 \). Find the amount of water that flows from the tank during the first 10 minutes. Simplify your answer. Show all work.

   (c)(4pts) Which limit below is equal to \( \int_{0}^{\pi} x \sin(x) \, dx \)? Choose only one answer. No justification necessary, copy down the entire answer. If you do not copy down the entire answer, points will be deducted.
(A) \( \lim_{n \to \infty} \sum_{i=1}^{n} \frac{\pi i}{n} \sin \left( \frac{\pi i}{n} \right) \)

(B) \( \lim_{n \to \infty} \sum_{i=1}^{n} \frac{\pi i}{n} \sin \left( \frac{\pi i}{n} \right) \frac{\pi}{n} \)

(C) \( \lim_{n \to \infty} \sum_{i=1}^{n} \sin \left( \frac{\pi i}{n} \right) \frac{\pi}{n} \)

(D) \( \lim_{n \to \infty} \sum_{i=1}^{n} \sin \left( \frac{\pi i}{n} \right) \frac{\pi^2 i}{n^2} \)

Solution:

(a) (12pts) Note that \( \Delta x = \frac{b - a}{n} = \frac{\pi - 0}{5} = \frac{\pi}{5} \) and \( x_i = a + \Delta x = 0 + \frac{\pi i}{5} = \frac{\pi i}{5} \) for \( i = 0, 1, \ldots, 5 \) and, since left endpoints have been specified, we have \( x^*_i = x_i \) thus

\[
\int_{0}^{\pi} x \sin(x) \, dx \approx L_5 = \frac{\pi}{5} \sum_{i=1}^{5} f(x_{i-1}) \Delta x = f(x_0) \Delta x + f(x_1) \Delta x + f(x_2) \Delta x + f(x_3) \Delta x + f(x_4) \Delta x
\]

\[
= 0 \cdot \sin \left( \frac{\pi}{5} \right) \cdot \frac{\pi}{5} + \frac{2 \pi}{5} \cdot \pi \cdot \frac{3 \pi}{5} \cdot \frac{\pi}{5} + \frac{4 \pi}{5} \cdot \frac{\pi}{5} \cdot \frac{\pi}{5} \cdot \frac{\pi}{5}
\]

\[
= \pi^2 \left[ \sin \left( \frac{\pi}{5} \right) + 2 \sin \left( \frac{\pi}{5} \right) + 3 \sin \left( \frac{\pi}{5} \right) + 4 \sin \left( \frac{\pi}{5} \right) \right]
\]

(b) (12pts) Here we have

\[
\int_{0}^{10} r(t) \, dt = \int_{0}^{10} (200 - 4t) \, dt = \left[ 200t - \frac{4t^2}{2} \right]_{0}^{10} = (200 \cdot 10 - 2 \cdot 10^2) - (200 \cdot 0 - 2 \cdot 0^2) = 2000 - 200 = 1800 \text{ liters}
\]

(c) (4pts) Choice D. Note that \( \Delta x = \frac{b - a}{n} = \frac{\pi}{n} \) and \( x_i = a + i \Delta x = \frac{\pi i}{n} \) and if we use right endpoints then \( x^*_i = x_i \) thus

\[
\int_{0}^{\pi} x \sin(x) \, dx = \frac{\pi}{5} \sum_{i=1}^{n} f(x_{i}) \Delta x = \frac{\pi}{5} \sum_{i=1}^{n} \frac{\pi i}{n} \sin \left( \frac{\pi i}{n} \right) \frac{\pi}{n} = \frac{\pi}{5} \sum_{i=1}^{n} \sin \left( \frac{\pi i}{n} \right) \frac{\pi^2 i}{n^2} \Rightarrow \text{Choice D.}
\]

Note that in Choice B, the limit is evaluated as \( i \to \infty \) which is incorrect.

3. (20pts) The following problems are not related.

(a) (10pts) Suppose we want to approximate the value of \( \sqrt{2} \) using Newton’s Method. What would the formula for \( x_{n+1} \) be? (To get full points for this question you must provide the explicit formula for \( x_{n+1} \) in terms of \( x_n \), the generic formula for Newton’s Method is not sufficient. You do not need to approximate the solution.)

(b) (10pts) Find the most general antiderivative of the function \( f(x) = \sqrt{x} + \sec^2(x) + \pi^2 \). Show all work.

Solution: (a) (10pts) Let \( x = \sqrt{2} = 2^{1/3} \) then \( x^3 = 2 \Rightarrow x^3 - 2 = 0 \). Thus, we wish to approximate the root of \( f(x) = x^3 - 2 \) using Newton’s Method. Note that \( f'(x) = 3x^2 \) thus

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 2}{3x_n^2} = \frac{3x_n^3 - (x_n^3 - 2)}{3x_n^2} = \frac{3x_n^3 - x_n^3 + 2}{3x_n^2} = \frac{2x_n^3 + 2}{3x_n^2} \text{ for } n = 1, 2, \ldots
\]

(b) (10pts) Note that

\[
\int \left( x^{1/3} + \sec^2(x) + \pi^2 \right) \, dx = \frac{x^{1/3 + 1}}{1/3 + 1} + \tan(x) + \pi^2 x + C = \frac{3}{4} x^{4/3} + \tan(x) + \pi^2 x + C
\]

4. (28pts) The following problems are not related.

(a) (12pts) If \( \int_{0}^{1} x \sqrt{x^2 + 4} \, dx = 5\sqrt{5} - 8 \), find \( \int_{1}^{0} \left[ t \sqrt{t^2 + 4} - 10 \right] \, dt \). Show all work.

(b) (12pts) Evaluate the definite integral \( \int_{0}^{3} |x - 2| \, dx \). Show all work. Simplify your answer.

(c) (4pts) Suppose, for the function \( f(x) \), we have \( f'(x) = \int \frac{6(3x^2 + 1)}{(x^2 - 1)^3} \, dx \). Which graph below best matches the graph of the function \( f(x) \)? (Hint: Do not try to solve the integral, you already have all the information needed to solve this
problem.) Choose only **one** answer. No justification necessary, clearly indicate your answer otherwise points will be deducted.

Solution: (a)(12pts) Note that, by changing the “dummy variable”, we have \[ \int_0^1 t \sqrt{t^2 + 4} \, dt = \int_0^1 x \sqrt{x^2 + 4} \, dx = 5\sqrt{5} - 8 \] thus
\[ \int_0^1 \left[ t \sqrt{t^2 + 4} - 10 \right] \, dt = - \int_0^1 \left[ t \sqrt{t^2 + 4} - 10 \right] \, dt = - \int_0^1 t \sqrt{t^2 + 4} \, dt + \int_0^1 10 \, dt = - \left( 5\sqrt{5} - 8 \right) + 10 \left|_0^1 \right. = -5\sqrt{5} + 8 + 10 = 18 - 5\sqrt{5} .\]

(b)(12pts) This problem can be done geometrically or directly. Note that the graph of \( f(x) = |x - 2| \) looks like: and so we see the region of interest can be described by two triangles, one triangle with base and height length of 2 units and the other triangle with base and height length of 1 unit, thus
\[ \int_0^3 |x - 2| \, dx = \int_0^2 |x - 2| \, dx + \int_2^3 |x - 2| \, dx = \int_0^2 -(x - 2) \, dx + \int_2^3 (x - 2) \, dx \]
thus,
\[ \int_0^3 |x - 2| \, dx = \int_0^2 -(x - 2) \, dx + \int_2^3 (x - 2) \, dx = \left. -\left( \frac{x^2}{2} - 2x \right) \right|_0^2 + \left. \left( \frac{x^2}{2} - 2x \right) \right|_2^3 = \left. -\left( \frac{2^2}{2} - 2 \cdot 2 \right) + 0 + \left( \frac{3^2}{2} - 2 \cdot 3 \right) - \left( \frac{2^2}{2} - 2 \cdot 2 \right) \right|_0^2 = 2 + \frac{9}{2} - 6 + 2 = \frac{9}{2} - 2 = \frac{5}{2} .\]

(c)(4pts) **Choice C.** Note that \( f'(x) = \int \frac{6(3x^2 + 1)}{(x^2 - 1)^3} \, dx \) implies that \( f'(x) \) is an antiderivative of \( \frac{6(3x^2 + 1)}{(x^2 - 1)^3} \) and so
\[ f''(x) = \frac{6(3x^2 + 1)}{(x^2 - 1)^3} \]
thus we can determine the concavity of the curve \( f(x) \). Note

\[ \begin{array}{c c c}
\text{Concavity of } f(x) \\
& + & - & + \\
& -1 & & 1 \\
\end{array} \]

Sign chart for \( f''(x) = \frac{6(3x^2 + 1)}{(x^2 - 1)^3} \)

and choice C best fits this concavity pattern.