1. (28pts) The following problems are not related.

(a)(12pts) Use limits to find all *vertical asymptotes* of $f(x) = \dfrac{x^2-4}{x^2+5x-14}$. Justify your answer with limits.

(b)(12pts) Use the Quotient Rule to find the derivative of $g(x) = \dfrac{2x+1}{x^2-1}$. Simpify your answer, show all work.

(c)(4pts) The function $h(x) = \sqrt{1-x^2}$ has a *vertical tangent* at which choice below? **Choose only <u>one</u> answer, no justification necessary, copy down the entire answer. If you do not copy down the entire answer, points will be deducted.**

(A) $x=-1,0,1$ (B) $x=-1,1$ (C) $x=\frac{1}{2}$ (D) $-1 \le x \le 1$ (E) None of these

Solution: (a)(12pts) Factoring yields

$$\frac{x^2-4}{x^2+5x-14} = \frac{(x-2)(x+2)}{(x-2)(x+7)} = \frac{x+2}{x+7}$$

so we have a removable discontinuity at $x=2$ and note that

$$\lim_{x\to -7^-} \frac{x^2-4}{x^2+5x-14} = \lim_{x\to -7^-} \frac{x+2}{x+7} = +\infty \text{ and } \lim_{x\to -7^+} \frac{x^2-4}{x^2+5x-14} = \lim_{x\to -7^+} \frac{x+2}{x+7} = -\infty$$

thus we have $\boxed{\text{a vertical asymptote at } x=-7.}$

(b)(12pts) By the Quotient Rule,

$$\frac{d}{dx}\left[\frac{2x+1}{x^2-1}\right] = \frac{2\cdot(x^2-1)-(2x+1)\cdot 2x}{(x^2-1)^2} = \frac{(2x^2-2)-(4x^2+2x)}{(x^2-1)^2} = \frac{-2x^2-2x-2}{(x^2-1)^2} = \boxed{\frac{-2(x^2+x+1)}{(x^2-1)^2}}$$

(c)(4pts) $\boxed{\text{Choice B.}}$ Note that

$$h'(x) = \frac{1}{2}(1-x^2)^{-1/2}\cdot(-2x) = \frac{-x}{\sqrt{1-x^2}}$$

and

$$\lim_{x\to -1^+} \frac{-x}{\sqrt{1-x^2}} = +\infty \text{ and } \lim_{x\to 1^-} \frac{-x}{\sqrt{1-x^2}} = -\infty$$

thus $h(x)$ has vertical tangents at $x=\pm 1$.

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2. (24pts) The following problems are not related.

(a)(12pts) The position function of a particle (in meters) at time t seconds is given by $s(t) = t^3 - 4.5t^2 - 7t$, $t \ge 0$. When does the particle reach a velocity of 5 meters/sec? Show all work.

(b)(12pts) Chip flies a kite at a constant height of 150 ft above the ground with the wind carrying the kite horizontally away from Chip at a rate of 25 ft/sec. How fast must Chip let out the string when 250 ft of string is out? Simplify your answer.

Solution: (a)(12pts) Note that $v(t) = s'(t) = 3t^2 - 9t - 7$ and

$$v(t) = 5 \Rightarrow 3t^2-9t-7=5 \Rightarrow 3t^2-9t-12=0 \Rightarrow 3(t^2-3t-4) \Rightarrow 3(t+1)(t-4)=0 \Rightarrow t=-1,4$$

thus the particle reaches a velocity of 5 m/sec when $\boxed{t = 4\,\text{seconds.}}$

(b)(12pts) The kite string, z, forms the hypotenuse of a right triangle. We are given $\frac{dx}{dt} = 25\,\text{ft/sec}$ and wish to find $\frac{dz}{dt}$ when $z = 250\,\text{ft}$. Note that

$$z^2 = x^2 + 150^2 \Rightarrow 2z\frac{dz}{dt} = 2x\frac{dx}{dt} \Rightarrow \frac{dz}{dt} = \frac{x}{z}\frac{dx}{dt}$$

and when $z = 250$ we have a "3-4-5" right triangle and so $x = 4 \cdot 50 = 200$, that is,

$$250^2 = x^2 + 150^2 \Rightarrow (5 \cdot 50)^2 = x^2 + (3 \cdot 50)^2 \Rightarrow x^2 = (4 \cdot 50)^2 \Rightarrow x = 4 \cdot 50 = 200.$$

Thus

$$\frac{dz}{dt} = \frac{x}{z}\frac{dx}{dt} = \frac{200}{250} \cdot 25 = 20\,\text{ft/sec}$$

so Chip must let the string out at $\boxed{20\,\text{ft/sec.}}$

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3. (28pts) The following problems are not related.

(a)(12pts) The radius of a circular disk is given as 12 cm with a maximum error in measurement of +0.2 cm. Use differentials to estimate the maximum error in the calculated area of the disk. Show all work. Simplify your answer, you may give your answer in terms of π.

(b)(12pts) Find dy/dx by implicit differentiation given that $\cos(xy) = 1 + \sin(y)$. Simplify your answer.

(c)(4pts) The function $f(x) = \dfrac{3x+1}{x + \sqrt{4x^2+5}}$ has a *horizontal asymptote* at which choice below? **Choose only one answer, no justification necessary, copy down the entire answer. If you do not copy down the entire answer, points will be deducted.**

(A) $y=3$ (B) $y=1$ (C) $y=3$ and $y=1$ (D) $y=-3$ and $y=1$ (E) None of these

Solution: (a)(12pts) Note that the area of a circle is $A = \pi r^2$ and $\Delta r = +0.2\,\text{cm}$ when $r = 12\,\text{cm}$ thus

$$A = \pi r^2 \Rightarrow dA = 2\pi r dr\Big|_{r=12, dr=\Delta r=0.2} = 2\pi \cdot 12 \cdot (0.2) = 4.8\pi\,\text{cm}^2$$

thus $\boxed{\text{the maximum error in the calculated area of the disk will be approximately } 4.8\pi\,\text{cm}^2.}$

(b)(12pts) Differentiating both sides with respect to x yields

$$-\sin(xy) \cdot [1 \cdot y + xy'] = 0 + \cos(y) \cdot y' \Rightarrow -y\sin(xy) = \big[\cos(y) + x\sin(xy)\big]y'$$

thus $\boxed{y' = \dfrac{-y\sin(xy)}{\cos(y) + x\sin(xy)}.}$

(c)(4pts) $\boxed{\text{Choice D.}}$ Note that

$$\lim_{x\to\infty} \frac{3x+1}{x+\sqrt{4x^2+5}} = \lim_{x\to\infty} \frac{x(3+1/x)}{x + 2|x|\sqrt{1+5/4x^2}} = \lim_{x\to\infty} \frac{x(3+1/x)}{x+2x\sqrt{1+5/4x^2}} = \lim_{x\to\infty} \frac{\cancel{x}(3+1/x)}{\cancel{x}(1+2\sqrt{1+5/4x^2})} = \frac{3}{1+2} = 1$$

thus $y = 1$ is a horizontal asymptote of $f(x)$ and, similarly,

$$\lim_{x\to-\infty} \frac{3x+1}{x+\sqrt{4x^2+5}} = \lim_{x\to-\infty} \frac{x(3+1/x)}{x + 2|x|\sqrt{1+5/4x^2}} = \lim_{x\to-\infty} \frac{x(3+1/x)}{x-2x\sqrt{1+5/4x^2}} = \lim_{x\to-\infty} \frac{\cancel{x}(3+1/x)}{\cancel{x}(1-2\sqrt{1+5/4x^2})} = \frac{3}{1-2} = -3.$$

thus $y = -3$ is also a horizontal asymptote of $f(x)$.

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4. (20pts) The following problems are not related.

(a)(12pts) Find the absolute minimum and maximum values of $f(x) = x\sqrt{3+2x}$ on the interval $[-1.5, 3]$. Give your answer in the form (x, y). Show all work, justify your answers and clearly label your answers.

(b)(8pts) If $f'(x) = \lim_{h\to 0} \dfrac{\sec(x+h) - \sec(x)}{h}$, find $f'(\pi/4)$. Justify your answer, show all work.

Solution:

(a)(12pts) Since $f(x)$ is continuous, we can use the Closed Interval Method. Now note, by the product rule, we have

$$\begin{aligned} f'(x) = \frac{d}{dx}\left[x \cdot (3+2x)^{1/2}\right] &= 1 \cdot (3+2x)^{1/2} + x \cdot \frac{1}{2}(3+2x)^{-1/2} \cdot 2 \\ &= (3+2x)^{1/2} + \frac{x}{(3+2x)^{1/2}} = \frac{(3+2x)+x}{\sqrt{3+2x}} = \frac{3+3x}{\sqrt{3+2x}} = \frac{3(1+x)}{\sqrt{3+2x}} \end{aligned}$$

so we have critical points when $f'(x) = 0 \Rightarrow x = -1$ and when $f'(x)$ is undefined, *i.e.* when $\sqrt{3+2x} = 0 \Rightarrow x = -3/2$. Now we need to check the value of $f(x)$ at the critical points and the endpoints:

$$\begin{aligned} f(-3/2) = f(-1.5) &= 0 \\ f(-1) &= -1 \Rightarrow \boxed{\text{abs. min at } (-1,-1) \text{ and abs. max at } (3,9).} \\ f(3) &= 9 \end{aligned}$$

(b)(8pts) Here we have

$$\lim_{h\to 0} \frac{\sec(x+h) - \sec(x)}{h} = f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \Rightarrow f(x) = \sec(x) \Rightarrow f'(x) = \sec(x)\tan(x)$$

thus $\boxed{f'(\pi/4) = \sec(\pi/4)\tan(\pi/4) = \sqrt{2} \cdot 1 = \sqrt{2}.}$