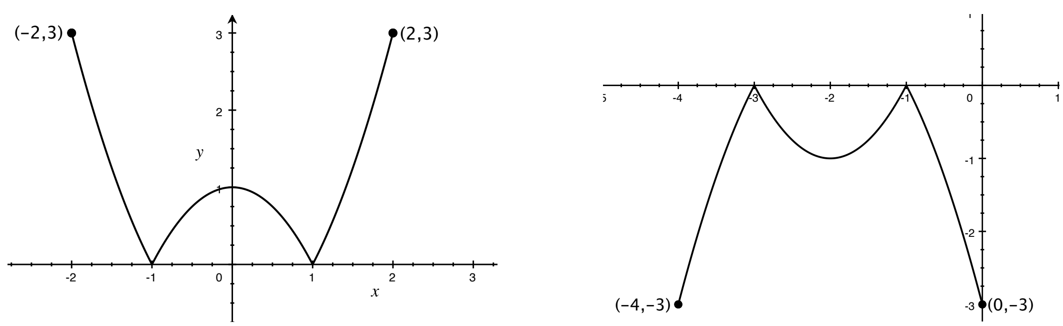
1. (28pts) The following problems are not related.

(a)(12pts) Suppose $f(x) = \dfrac{1}{\sqrt{4-x^2}}$ and $g(x) = \sqrt{x+3}$. Find $(f \circ g)(x)$ and express the domain of this function in interval notation.

(b)(12pts) Suppose that $h(x) = \begin{cases} \dfrac{x^2-4}{|x-2|}, & \text{if } x < 2 \\ \dfrac{\sqrt{2x}-10}{2}, & \text{if } x > 2 \end{cases}$, find the two-sided limit $\lim_{x\to 2} h(x)$. Show all work and justify your answer.

(c)(4pts) Consider the graph of the function below labeled as A. If this function is $y = f(x)$ then which of the following choices given below correctly represents the graph labeled as B? **No justification necessary**- *Choose only* <u>*one*</u> *answer, copy down the entire answer.*)

(A) $y = -f(x) - 2$ (B) $y = f(-x) - 2$ (C) $y = -f(x+2)$ (D) $y = f(-x+2)$ (E) $y = f(-x) + 2$



(a) Graph A - the function $y = f(x)$ (b) Graph B

Figure 1: Graphs for Problem 1(c)

Solution: (a)(12pts) Note that

$$(f \circ g)(x) = f(g(x)) = \underbrace{f(\sqrt{x+3})}_{\text{need } x \geq -3} = \frac{1}{\sqrt{4-(\sqrt{x+3})^2}} = \underbrace{\frac{1}{\sqrt{1-x}}}_{\text{need } x<1} \text{ with domain } [-3, 1).$$

(b)(12pts) We need to examine the 1-sided limits:

$$\lim_{x\to 2^-} h(x) = \lim_{x\to 2^-} \frac{x^2-4}{|x-2|} = \lim_{x\to 2^-} \frac{x^2-4}{-(x-2)} = \lim_{x\to 2^-} \frac{(x-2)(x+2)}{-(x-2)} = \lim_{x\to 2^-} \frac{(x+2)}{-1} = -4$$

and

$$\lim_{x\to 2^+} h(x) = \lim_{x\to 2^+} \frac{\sqrt{2x}-10}{2} = \frac{\sqrt{4}-10}{2} = \frac{-8}{2} = -4 \text{ thus } \boxed{\lim_{x\to 2} h(x) = -4.}$$

(c)(4pts) $\boxed{\text{Choice C.}}$ If we shift Graph A to the left 2 units and then reflect it about the x-axis this yields Graph B.

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2. (24pts) The following problems are not related.

(a)(12pts) Suppose $4x + x^2 \leq 2g(x) + x^2 \leq 2x^4 - x^2 + 4$ for all x near 1, find $\lim_{x\to 1} g(x)$. Show all work, explain your answer. Clearly state any named theorems you are using.

(b)(12pts) Find the limit $\lim_{x\to 0} \dfrac{\sin(3x)\sin(5x)}{x^2}$. Justify your answer, show all work.

Solution: (a)(12pts) Note that

$$4x + x^2 \le 2g(x) + x^2 \le 2x^4 - x^2 + 4 \Longrightarrow 4x \le 2g(x) \le 2x^4 - 2x^2 + 4 \Longrightarrow 2x \le g(x) \le x^4 - x^2 + 2$$

and since $\lim_{x\to 1} 2x = \lim_{x\to 1} x^4 - x^2 + 2 = 2$ thus, $\boxed{\text{by } \underline{\text{Squeeze Theorem}},\ \lim_{x\to 1} g(x) = 2.}$

(b)(12pts) Using the special limit $\lim_{\theta\to 0} \dfrac{\sin(\theta)}{\theta} = 1$ we have

$$\lim_{x\to 0} \frac{\sin(3x)\sin(5x)}{x^2} = \lim_{x\to 0} \frac{\sin(3x)}{x} \cdot \frac{\sin(5x)}{x} = \lim_{x\to 0} \frac{\sin(3x)}{3x} \cdot \frac{\sin(5x)}{5x} \cdot 3 \cdot 5 = 1 \cdot 1 \cdot 15 = \boxed{15}$$

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3. (28pts) The following problems are not related.

(a)(12pts) Evaluate the limit: $\lim_{x\to 0} \dfrac{\sqrt{3+x} - \sqrt{3}}{x}$. Show all work.

(b)(12pts) Consider the function $f(x) = \dfrac{x^3 - x^2 - 2x}{(x+3)(x-2)}$. (i)(9pts) Find the $\lim_{x\to 2} f(x)$. (ii)(3pts) Is $f(x)$ continuous at $x = 2$? If not, what type of discontinuity is at $x = 2$? Justify your answers.

(c)(4pts) If $\lim_{x\to a} f(x) = 2$ and $\lim_{x\to a} g(x) = 6$ then which of the choices below is equal to $\lim_{x\to a} [(f(x))^2 - 2f(x)g(x) + (g(x))^2]$? (**No justification necessary** - *Choose only* <u>*one*</u> *answer, copy down the entire answer.*)

(A) 40 (B) -4 (C) 16 (D) 28 (E) None of these

Solution: (a)(12pts) Multiplying by the conjugate yields

$$\lim_{x\to 0} \frac{\sqrt{3+x} - \sqrt{3}}{x} = \lim_{x\to 0} \frac{\sqrt{3+x} - \sqrt{3}}{x} \cdot \frac{\sqrt{3+x} + \sqrt{3}}{\sqrt{3+x} + \sqrt{3}} = \lim_{x\to 0} \frac{(3+x) - 3}{x(\sqrt{3+x} + \sqrt{3})} = \lim_{x\to 0} \frac{1}{\sqrt{3+x} + \sqrt{3}} = \boxed{\frac{1}{2\sqrt{3}}.}$$

(b)(i)(9pts) Factoring yields

$$\lim_{x\to 2} \frac{x^3 - x^2 - 2x}{(x+3)(x-2)} = \lim_{x\to 2} \frac{x(x^2 - x - 2)}{(x+3)(x-2)} = \lim_{x\to 2} \frac{x(x+1)(x-2)}{(x+3)(x-2)} = \lim_{x\to 2} \frac{x(x+1)}{(x+3)} = \boxed{\frac{6}{5}.}$$

(b)(ii)(3pts) $\boxed{\text{Removable discontinuity at } x = 2}$ since $f(2) = 0/0$ (the function does not exist at $x = 2$) but $\lim_{x\to 2} f(x) = 6/5$, *i.e.* the limit exists so the discontinuity at $x = 2$ is of the removable type.

(c)(4pts) $\boxed{\text{Choice C.}}$ Using the property of limits

$$\lim_{x\to a} [(f(x))^2 - 2f(x)g(x) + (g(x))^2] = [(\lim_{x\to a} f(x))^2 - 2 \cdot \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) + (\lim_{x\to a} g(x))^2] = 2^2 - 2 \cdot 2 \cdot 6 + 6^2 = 16.$$

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4. (20pts) The following problems are not related.

(a)(12pts) Find the real number a so that the function $f(x) = \begin{cases} \dfrac{3\sin(x)}{x}, & \text{if } x \neq 1 \\ ax + 8, & \text{if } x = 1 \end{cases}$ is continuous at $x = 1$. Explain your answer.

(b)(8pts) Use the Intermediate Value Theorem to show that the x-coordinate of at least one of the intersection points where the cubic curve $y = x^3 - 3x$ crosses the line $y = 1$ is in the interval $[0, 2]$. Explain your answer and be sure to verify that the requirements of the Intermediate Value Theorem have been satisfied.

Solution: (a)(12pts) For continuity at $x = 1$, we need $\lim_{x\to 1} f(x) = f(1)$ and note that $f(1) = a + 8$ and

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{3\sin(x)}{x} = \frac{3\sin(1)}{1} = 3\sin(1) \text{ so } a + 8 = 3\sin(1) \text{ implies we need } \boxed{a = 3\sin(1) - 8.}$$

(b)(8pts) We wish to show that the equation $x^3 - 3x = 1$ has a solution in $[0.2]$. Note that if $f(x) = x^3 - 3x$ then since $f(x)$ is $\boxed{\text{a polynomial it is continuous for all values of } x}$ and further note that $f(0) = 0 < 1$ and $f(2) = 8 - 6 = 2 > 1$, *i.e.* $\boxed{f(0) < 1 < f(2)}$ thus, by the Intermediate Value Theorem there exists a number c in the interval $(0, 2)$ such that $f(c) = 1 \Rightarrow c^3 - 3c = 1$.
