1. (24 points) The following problems are unrelated.

   (a) Find the derivative of $y = \sqrt{x^5} + \dfrac{4x^8}{3ax^{2/3}}$ where a is a constant. (Fully simplify your final answer.)

   (b) Find $s'(r)$ when $s(r) = \dfrac{5\cos(r) - 8r^2}{(7r^3+4)^{102}}$. (Please DO NOT simplify your final answer.)

   (c) Consider the function $h(x) = \sqrt{5-x}$.

      i. Determine the linearization, $L(x)$, of $y = h(x)$ at $a = 1$.

      ii. Use your linearization from (i) to approximate $\sqrt{4.1}$.

Solution:

(a) Note that $y = x^{5/2} + \dfrac{4}{3a}x^{22/3}$. So, we have $y' = \frac{5}{2}x^{3/2} + \frac{88}{9a}x^{19/3}$.

(b)

$$s'(r) = \frac{(7r^3+4)^{102}(-5\sin(r) - 16r) - (5\cos(r) - 8r^2)\cdot 102(7r^3+4)^{101}\cdot 21r^2}{(7r^3+4)^{204}}$$

(c) i. Note that $h(1) = \sqrt{5-1} = 2$. And since $h'(x) = \frac{-1}{2\sqrt{5-x}}$, then we have $h'(1) = -1/4$. So, the linearization is $L(x) = 2 - \frac{1}{4}(x-1)$.

ii.

$$\begin{aligned}\sqrt{4.1} &= \sqrt{5 - 9/10}\\ &= h(9/10)\\ &\approx L(9/10)\\ &= 2 - (1/4)(9/10 - 1)\\ &= 81/40\end{aligned}$$

That is, $\sqrt{4.1} \approx 81/40$.

2. (14 points) Consider the curve given by $x^2 + 4xy^2 + y^2 = 1$.

   (a) Find the value of y' at the point $(-4, 1)$.

   (b) Determine the normal line to the curve at $(-4, 1)$.

Solution:

(a)

$$\begin{aligned}\frac{d}{dx}\left(x^2 + 4xy^2 + y^2\right) &= \frac{d}{dx}(1)\\ 2x + 4y^2 + 8xyy' + 2yy' &= 0\\ (8x+2)yy' &= -(2x + 4y^2)\\ y' &= -\frac{x + 2y^2}{(4x+1)y}\end{aligned}$$

At that point, we have $y' = -2/15$.

(b) The slope of normal line will be $15/2$. So, the equation of the normal line is

$$y = 1 + \frac{15}{2}(x+4).$$

3. (15 pts) Consider $k(x) = x^5(x-3)^7$.

   (a) Determine the critical numbers of $k(x)$.

   (b) For each of the critical numbers you found in (a), determine if it is the location of a "local maximum," "local minimum," or "neither." Clearly state an answer for each critical number, and be sure to justify your answers using a theorem from this class.

Solution:

(a) We differentiate to find

$$\begin{aligned} k'(x) &= 5x^4(x-3)^7 + 7x^5(x-3)^6 \\ &= x^4(x-3)^6[5(x-3)+7x] \\ &= 3x^4(x-3)^6(4x-5). \end{aligned}$$

Since we have factored, it is short work to see that $k'(x) = 0$ when $x = 0, 5/4$, and 3. Since $k'(x)$ exists for all real numbers x, we see that the critical numbers of $k(x)$ are $x = 0, 5/4$, and 3.

(b) We have that $k'(x) < 0$ for $x < 5/4$ and $k'(x) > 0$ for $x > 5/4$. So the First Derivative Test implies that $k(x)$ has one local extreme value, which is a local minimum, at $x = 5/4$. (In fact, this is an absolute minimum!) $x = 0$ and $x = 3$ are neither the location of a local maximum nor a local minimum.

4. (20 points) Consider $g(x) = \dfrac{x}{2x+4}$.

   (a) Use the definition of the derivative to show $g'(x) = \dfrac{4}{(2x+4)^2}$. (Note: You must use the definition of the derivative to earn any credit on (a).)

   (b) Determine the average rate of change of g over $[-3, 0]$.

   (c) Does $g'(x)$ ever equal the value you obtained in (b)? (Justify your answer.)

   (d) Why does your answer in (c) not contradict the mean value theorem?

Solution:

(a)

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{x+h}{2x+2h+4} - \frac{x}{2x+4}}{h} \\ &= \lim_{h\to 0} \frac{(x+h)(2x+4) - x(2x+2h+4)}{h(2x+2h+4)(2x+4)} \\ &= \lim_{h\to 0} \frac{4}{(2x+2h+4)(2x+4)} \\ &= \frac{4}{(2x+4)^2}. \end{aligned}$$

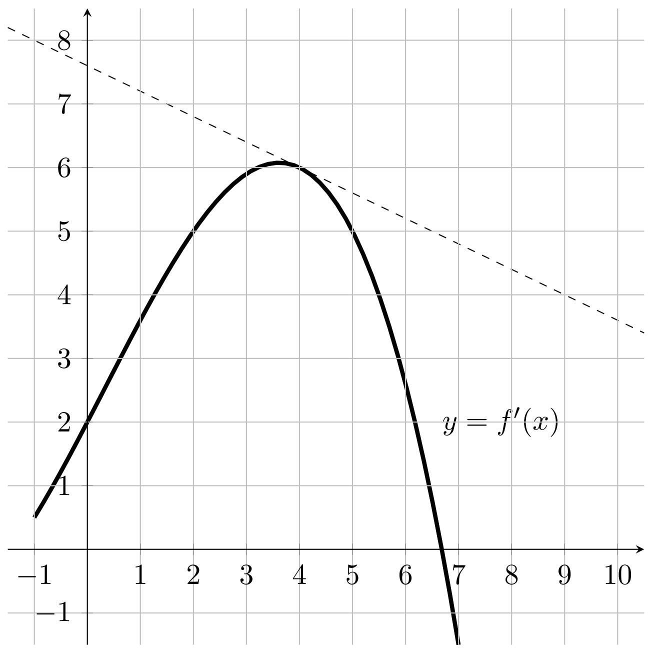
(b)

$$\frac{g(0)-g(-3)}{0-(-3)}=\frac{0-3/2}{3}=-1/2.$$

(c) We have that $g'(x)$ never equals $-1/2$ because $g'(x)>0$ for all x for which it is defined.

(d) This **does not** contradict the Mean Value Theorem because g is neither continuous on $[-3,0]$ nor differentiable on $(-3,0)$.

5. (15 points) The function $f(x)$ has domain $[-1,7]$. Consider the graph of $y=f'(x)$ below. The dashed line is the tangent line of $y=f'(x)$ at $x=4$. Use this graph to answer the questions that follow. No justifications are required on this problem.



(Remember that the graph above is the graph of $y=f'(x)$, not $y=f(x)$.)

(a) Evaluate $\lim\limits_{h\to 0}\dfrac{f'(4+h)-f'(4)}{h}$.

(b) On the interval $(5,6)$, is $y=f(x)$ increasing or decreasing?

(c) On the interval $(2,3)$, is $y=f(x)$ concave up or concave down?

(d) On which of the following intervals does $f(x)$ have a local extreme value: $(-1,1)$, $(3,5)$ or $(5,7)$?

(e) Is the local extreme value noted in (d) a maximum or a minimum?

(f) If $f(4)=2$, what is the equation of the tangent line of $y=f(x)$ at $x=4$?

Solution:

(a) The slope of the dashed line is $-2/5$.

(b) Increasing (because $f'(x)>0$)

(c) Concave up (because $f'(x)$ is increasing)

(d) $(5,7)$ (because f' changes from positive to negative)

(e) Maximum

(f) $y = 2 + 6(x - 4)$ (because from the graph we have $f'(4) = 6$)

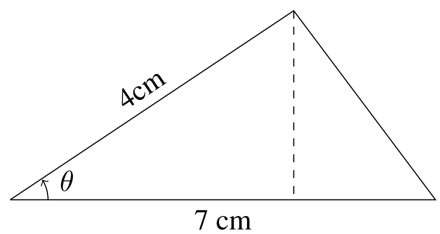
6. (12 points) Suppose that two sides of a triangle have fixed lengths of 4 cm and 7 cm, but that the angle between these two sides is growing at a rate of 3 radians per minute. Let θ denote the angle between these two sides of the triangle.

   (a) Draw this triangle and label the angle, θ, and the two sides of lengths 4 cm and 7 cm. (We recommend having the side of length 7 cm be the base of your triangle, as this may be helpful in (b).)

   (b) Determine the rate of change of the area of the triangle with respect to time when the angle is $\theta = \pi/4$. (Include the correct units in the final answer.)

Solution:

(a)



(b) We need to determine $\frac{dA}{dt}$. The height of our triangle is $4\sin\theta$, and its base is 7. So, the area of the triangle is $A = 14\sin\theta$, where A and θ both vary over time.

Differentiating, we have

$$\frac{dA}{dt} = 14\cos\theta \cdot \frac{d\theta}{dt}.$$

If we apply our given values for θ and $\frac{d\theta}{dt}$, then we have

$$\left.\frac{dA}{dt}\right|_{\theta=\pi/4} = 14\cos(\pi/4)\cdot 3 = 21\sqrt{2} \text{ square centimeters per minute.}$$