1. (16 points) Answer each of the following unrelated problems.

   (a) Given $\tan\theta = 10$ and $\csc\theta < 0$, what is $\cos\theta$?

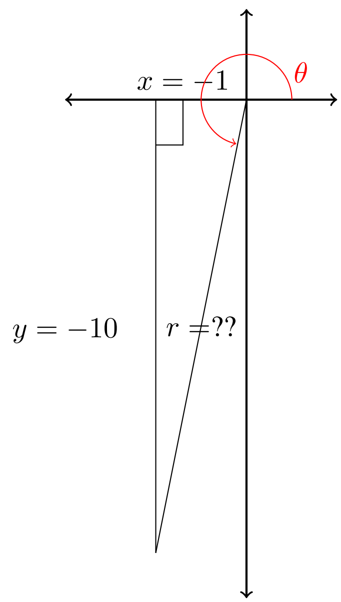
   (b) Find all values x in the interval $[0, 2\pi]$ that satisfy $\tan x \sec x = 4\sin x$.

Solution:

(a) Since $\tan\theta > 0$ but $\sin\theta = \frac{1}{\csc\theta} < 0$, then we know that θ is in the third quadrant, and that $\cos\theta < 0$.
Option 1:

$$\begin{aligned} \tan^2\theta + 1 &= \sec^2\theta \\ 100 + 1 &= \frac{1}{\cos^2\theta} \\ \cos\theta &= -\frac{1}{\sqrt{101}}. \end{aligned}$$

Option 2: We can draw the right triangle whose terminal side is in the third quadrant. Since $10 = \tan\theta = \frac{y}{x}$, we can use the values $x = -1$ and $y = -10$, as indicated below.



Using the Pythagorean theorem, we find $r = \sqrt{101}$. So, we have

$$\cos\theta = \frac{x}{r} = -\frac{1}{\sqrt{101}}.$$

(b)

$$\begin{aligned} \tan x \sec x &= 4\sin x \\ \frac{\sin x}{\cos^2 x} &= 4\sin x \\ \sin x\left(\frac{1}{\cos^2 x} - 4\right) &= 0. \end{aligned}$$

This implies $\sin x = 0$ or $\dfrac{1}{\cos^2 x} - 4 = 0$. The first gives solutions $x = 0, \pi$, and 2π. The second simplifies to $\cos x = \pm\frac{1}{2}$, which has solutions $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$, and $\frac{5\pi}{3}$. So, the solutions to the given equation are

$$x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, \text{ and } 2\pi.$$

2. (23 points) Evaluate the following limits and simplify your answers. If a limit does not exist, clearly state this. If you use a theorem, clearly state its name and show that its hypotheses are satisfied.
(Reminder: You may not use L'Hopital's Rule or Dominance of Powers in any solutions on this exam.)

   (a) $\lim\limits_{x\to 7} (5x + 1 - |x - 7|)$

   (b) $\lim\limits_{x\to 4} \dfrac{2-\sqrt{x}}{x^2 - 4x}$

   (c) $\lim\limits_{\theta\to\pi/4} \dfrac{\cos^2\theta - \sin^2\theta}{\cos\theta - \sin\theta}$

Solution:

   (a)

$$\lim_{x\to 7} (5x + 1 - |x - 7|) = 5(7) + 1 - |7 - 7| = 36.$$

   (b) **Option 1**

$$\begin{aligned}
\lim_{x\to 4} \frac{2-\sqrt{x}}{x^2 - 4x} &= \lim_{x\to 4} \frac{2-\sqrt{x}}{x(\sqrt{x}-2)(\sqrt{x}+2)} \\
&= \lim_{x\to 4} \frac{-1}{x(\sqrt{x}+2)} \\
&= \frac{-1}{4(\sqrt{4}+2)} \\
&= -\frac{1}{16}.
\end{aligned}$$

   Option 2

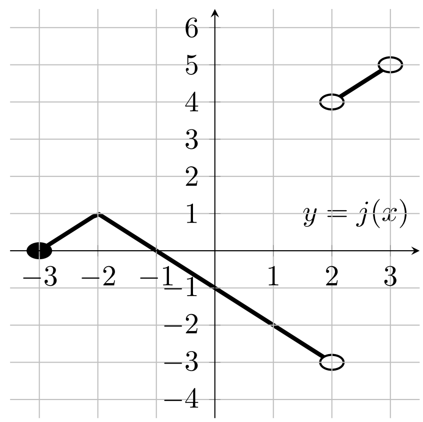
$$\begin{aligned}
\lim_{x\to 4} \frac{2-\sqrt{x}}{x^2 - 4x} &= \lim_{x\to 4} \frac{2-\sqrt{x}}{x(x-4)} \cdot \frac{2+\sqrt{x}}{2+\sqrt{x}} \\
&= \lim_{x\to 4} \frac{4-x}{x(x-4)(2+\sqrt{x})} \\
&= \lim_{x\to 4} \frac{-1}{x(\sqrt{x}+2)} \\
&= \frac{-1}{4(\sqrt{4}+2)} \\
&= -\frac{1}{16}.
\end{aligned}$$

   (c)

$$\lim_{\theta\to\pi/4} \frac{\cos^2\theta - \sin^2\theta}{\cos\theta - \sin\theta} = \lim_{\theta\to\pi/4} \frac{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}{\cos\theta - \sin\theta}$$

$$
\begin{aligned}
&= \lim_{\theta\to\pi/4} \cos\theta + \sin\theta \\
&= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \\
&= \sqrt{2}.
\end{aligned}
$$

3. (15 points) Use the graph of $y = j(x)$ provided here to answer each of the problems below. A brief justification is only required on (g). (Note that the entire graph of $y = j(x)$ is represented in the graph below.)



   (a) Find $\lim_{x\to -2} j(x)$, if it exists. (If it does not exist, state Does Not Exist.)

   (b) Find $\lim_{x\to 2^-} \sqrt{j(x)}$, if it exists. (If it does not exist, state Does Not Exist.)

   (c) Find $\lim_{x\to 2^+} \sqrt{j(x)}$, if it exists. (If it does not exist, state Does Not Exist.)

   (d) State the domain of $j(x)$ using interval notation.

   (e) State the range of $j(x)$ using interval notation.

   (f) Is $y = j(x)$ odd, even, or neither?

   (g) Note that $j(1) < 0$ and $j(2.5) > 0$. Does the Intermediate Value Theorem imply $j(c) = 0$ for some c in $[1, 2.5]$? Why or why not?

Solution:

   (a) 1

   (b) Does not exist

   (c) $\sqrt{4} = 2$

   (d) $[-3, 2) \cup (2, 3)$

   (e) $(-3, 1] \cup (4, 5)$

   (f) Neither

   (g) No because j is not continuous on $[1, 2.5]$.

4. (22 points) Consider the functions $f(x) = \dfrac{x^2 - 9}{x - 3}$ and $g(x) = \dfrac{|x^2 - 9|}{x - 3}$. Answer each of the following:

   (a) Identify all discontinuities of $f(x)$ as a removable discontinuity, a jump discontinuity, or an infinite discontinuity. For each, justify using appropriate limits.

(b) Write $g(x)$ as a piecewise defined function where each piece is a polynomial.

(c) Identify all discontinuities of $g(x)$ as a removable discontinuity, a jump discontinuity, or an infinite discontinuity. For each, justify using appropriate limits.

Solution:

(a) This function is rational, and therefore continuous on its domain. The only value not in its domain is $x = 3$. We note that.

$$\begin{aligned}\lim_{x\to 3} f(x) &= \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} \\ &= \lim_{x\to 3} x+3 \\ &= 6,\end{aligned}$$

but $f(3)$ does not exist. So, f has a removable discontinuity at $x = 3$.

(b) We note that $|x^2 - 9| = x^2 - 9$ if $x \geq 3$ or $x \leq -3$, and $|x^2 - 9| = -(x^2 - 9)$ if $-3 \leq x \leq 3$. We also see that this function is defined everywhere but 3. So,

$$g(x) = \begin{cases} x+3, & x > 3 \text{ or } x \leq -3 \\ -x-3, & -3 < x < 3 \end{cases}$$

.

We could instead say

$$g(x) = \begin{cases} x+3, & x > 3 \text{ or } x < -3 \\ -x-3, & -3 \leq x < 3 \end{cases}$$

.

(c) Note that there is no discontinuity at $x = -3$ since

$$\lim_{x\to -3^-} g(x) = \lim_{x\to -3^+} g(x) = g(-3) = 0.$$

At $x = 3$, we see that

$$\lim_{x\to 3^-} g(x) = \lim_{x\to 3^-} -x-3 = -6$$

but

$$\lim_{x\to 3^+} g(x) = \lim_{x\to 3^+} x+3 = 6.$$

So, g has a jump discontinuity at $x = 3$.

5. (24 points) Consider the function $h(x) = \dfrac{\sqrt{4x^2-2}}{x+6}$.

(a) State the domain of $h(x)$ using interval notation.

(b) Determine all vertical asymptotes of $y = h(x)$ and justify each with the appropriate limit.

(c) Determine all horizontal asymptotes of $y = h(x)$ and justify each with the appropriate limit.

Solution:

(a) We note that $x \neq -6$ and that $4x^2 - 2 \geq 0$. We need to solve that latter inequality. It simplifies to $x^2 \geq \frac{1}{2}$, which tells us $x \geq \frac{1}{\sqrt{2}}$ or $x \leq -\frac{1}{\sqrt{2}}$. So, the domain is

$$(-\infty, -6) \cup \left(-6, -\frac{1}{\sqrt{2}}\right] \cup \left[\frac{1}{\sqrt{2}}, \infty\right).$$

(b) The only possible location of a vertical asymptote is at $x = -6$, because h is an algebraic function that has division by 0 only at that value.

We note that

$$\lim_{x \to -6^+} h(x) = \lim_{x \to -6^+} \frac{\sqrt{4x^2 - 2}}{x + 6} = \infty$$

because $\sqrt{4x^2 - 2} \to \sqrt{4 \cdot 36 - 2} > 0$ and $x + 6 \to 0^+$ as $x \to -6^+$. This shows that $y = h(x)$ has a vertical asymptote at $x = -6$. (We could have alternatively have shown $\lim\limits_{x \to -6^-} h(x) = -\infty$ for our justification.)

(c) We need to check both limits at infinity. Note that both of these limits are $\frac{\infty}{\infty}$-indeterminate forms.

$$\begin{aligned}
\lim_{x \to \infty} h(x) &= \lim_{x \to \infty} \frac{\sqrt{4x^2 - 2}}{x + 6} \\
&= \lim_{x \to \infty} \frac{\sqrt{x^2 \left(4 - \frac{2}{x^2}\right)}}{x \left(1 + \frac{6}{x}\right)} \\
&= \lim_{x \to \infty} \frac{|x| \sqrt{4 - \frac{2}{x^2}}}{x \left(1 + \frac{6}{x}\right)} \\
&= \lim_{x \to \infty} \frac{\sqrt{4 - \frac{2}{x^2}}}{1 + \frac{6}{x}} \\
&= \frac{\sqrt{4 - 0}}{1 + 0} \\
&= 2,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{x \to -\infty} h(x) &= \lim_{x \to -\infty} \frac{\sqrt{4x^2 - 2}}{x + 6} \\
&= \lim_{x \to -\infty} \frac{\sqrt{x^2 \left(4 - \frac{2}{x^2}\right)}}{x \left(1 + \frac{6}{x}\right)} \\
&= \lim_{x \to -\infty} \frac{|x| \sqrt{4 - \frac{2}{x^2}}}{x \left(1 + \frac{6}{x}\right)} \\
&= \lim_{x \to -\infty} \frac{-\sqrt{4 - \frac{2}{x^2}}}{1 + \frac{6}{x}} \\
&= \frac{-\sqrt{4 - 0}}{1 + 0} \\
&= -2.
\end{aligned}$$

So, $y = h(x)$ has horizontal asymptotes at $y = \pm 2$.