1. (40 pts) The following problems are unrelated.

   (a) Find the equation of the tangent line of $y = \ln\left(x^2 \ln x\right)$ at $x = e$.

   (b) Evaluate $\displaystyle\int_{-\frac{\ln(2)}{3}}^{-\frac{\ln(2/\sqrt{3})}{3}} \frac{1}{e^{-3x}\sqrt{1-e^{6x}}}\,dx$. (Just to be sure it's clear, the lower limit of integration is $-\dfrac{\ln(2)}{3}$ and the upper limit is $-\dfrac{\ln(2/\sqrt{3})}{3}$.)

   (c) Evaluate $\displaystyle\lim_{x\to\infty} \frac{\log_4(x+6)}{\log_3(x)}$.

   (d) Evaluate $\sin^{-1}\left(\sin(3\pi/4)\right) - \tan^{-1}\left(\tan(4\pi/5)\right)$. Fully simplify your answer.

Solution:

(a) We need the point and the slope. Note that $y(e) = \ln(e^2 \cdot 1) = 2$. We have derivative

$$y' = \frac{x + 2x\ln(x)}{x^2 \ln x} = \frac{1 + 2\ln(x)}{x \ln x}.$$

This gives $y'(e) = \frac{e+2e\cdot 1}{e^2\cdot 1} = 3e^{-1}$. Thus, the tangent line is

$$y = 2 + 3e^{-1}(x-e).$$

(b) We will apply the substitution $u = e^{3x}$. Then, $du = 3e^{3x}\,dx$, the lower limit of integration becomes

$$u = e^{-\ln 2} = e^{\ln(1/2)} = \frac{1}{2}$$

and the upper limit of integration becomes

$$u = e^{-\ln(2/\sqrt{3})} = e^{\ln(\sqrt{3}/2)} = \frac{\sqrt{3}}{2}$$

So, we have

$$\begin{aligned}\int_{-\frac{\ln(2)}{3}}^{-\frac{\ln(2/\sqrt{3})}{3}} \frac{e^{3x}}{\sqrt{1-e^{6x}}}\,dx &= \frac{1}{3}\int_{1/2}^{\sqrt{3}/2} \frac{1}{\sqrt{1-u^2}}\,du \\ &= \frac{1}{3}\left(\arcsin(\sqrt{3}/2) - \arcsin(1/2)\right) \\ &= \frac{\pi}{18}.\end{aligned}$$

(c) This limit is an $\frac{\infty}{\infty}$-indeterminate form, so we will begin by applying L'Hospital's Rule:

$$\begin{aligned}\lim_{x\to\infty} \frac{\log_4(x+6)}{\log_3(x)} &= \lim_{x\to\infty} \frac{\ln(3)x}{\ln(4)(x+6)} \cdot \frac{1/x}{1/x} \\ &= \lim_{x\to\infty} \frac{\ln 3}{\ln 4(1+6/x)} \\ &= \frac{\ln 3}{\ln 4}.\end{aligned}$$

(d) We first consider the first term. Note that $\sin(3\pi/4) = 1/\sqrt{2}$. So, we have $\sin^{-1}(\sin(3\pi/4)) = \sin^{-1}\left(1/\sqrt{2}\right) = \pi/4$.

Now, let us consider the second term. Let $\tan(4\pi/5) = b/a$. The quotient b/a must be negative since $4\pi/5$ is in Quadrant 2. Thus, $\tan^{-1}(\tan(4\pi/5)) = \tan^{-1}(b/a))$ must be a number on the interval $(-\pi/2, \pi/2)$, per the definition of the arctan function. Furthermore, since $b/a < 0$, $\tan^{-1}(b/a))$ must be a number on the interval $(-\pi/2, 0)$. Because the reference angle for $4\pi/5$ is $\pi/5$, $\tan^{-1}(\tan(4\pi/5))$ must equal $-\pi/5$.

Therefore, $\sin^{-1}(\sin(3\pi/4)) - \tan^{-1}(\tan(4\pi/5)) = \pi/4 - (-\pi/5) = \boxed{9\pi/20}$

2. (16 pts) Let $f(x) = \begin{cases} x^2 - x & x \geq 0 \\ |x| & x < 0 \end{cases}$

(a) Find $f'(x)$. Your answer will also be a piecewise-defined function. Specifically, do the following:
- Use any tools from our class to find $f'(x)$ when $x \neq 0$.
- Use the definition of the derivative to find $f'(0)$ or to argue that $f'(0)$ does not exist.

(b) Use your answer from (a) and the definition of the derivative to find $f''(0)$, or to argue that it does not exist.

Solution:

(a) When $x < 0$ we have $f(x) = -x$ and the derivative is $f'(x) = -1$. When $x > 0$ we have $f(x) = x^2 - x$ and the derivative is $f'(x) = 2x - 1$. When $x = 0$ the limit as $h \to 0$ from both sides must be equal for the derivative to exist:

$$\lim_{h\to 0^-} \frac{-(0+h)+0}{h} = \lim_{h\to 0} \frac{-h}{h} = -1, \text{ and}$$

$$\lim_{h\to 0^+} \frac{(0+h)^2 - (0+h) - (0^2 - 0)}{h} = \lim_{h\to 0^+} h - 1 = -1.$$

Since both limits have the same value, $f'(0) = -1$ and we have

$$f'(x) = \begin{cases} 2x - 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

(b) We consider the limit $\lim\limits_{h\to 0} \dfrac{f'(0+h) - f'(0)}{h}$. We must take the limit as $h \to 0$ from both sides to determine if this limit exists. When $x = 0$ and $h \to 0^-$, we have $f'(h) = -1$ and $f'(0) = -1$. The limit is

$$\lim_{h\to 0^-} \frac{-1-(-1)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

When $x = 0$ and $h \to 0^+$, we have $f'(h) = 2h - 1$ and $f'(0) = -1$. The limit is

$$\lim_{h\to 0^+} \frac{2h - 1 - (-1)}{h} = \lim_{h\to 0^+} \frac{2h}{h} = 2.$$

Since these limits are not equal, $f''(0)$ does not exist.

3. (26 pts) Consider the function $f(x) = ax + \ln(\cos(x))$ on the domain $[0, \pi/2)$, where $a > 0$ is a constant.

(a) Is $f(x)$ concave up anywhere on this domain? If so, give the interval(s) where that occurs.

(b) Find the x-coordinate of the absolute maximum of $f(x)$ on the given domain. Note: your answer will be in terms of a. Be sure to justify that you have located the absolute maximum.

(c) Now, set $a = 1$.

(i) Determine if the Mean Value Theorem applies to $f(x)$ on $[0, \pi/4]$. (Clearly note if the hypotheses of the theorem apply *or* do not apply here.)

(ii) If the answer to (i) is yes, find the value of c that satisfies the conclusion of the Mean Value Theorem.

Solution:

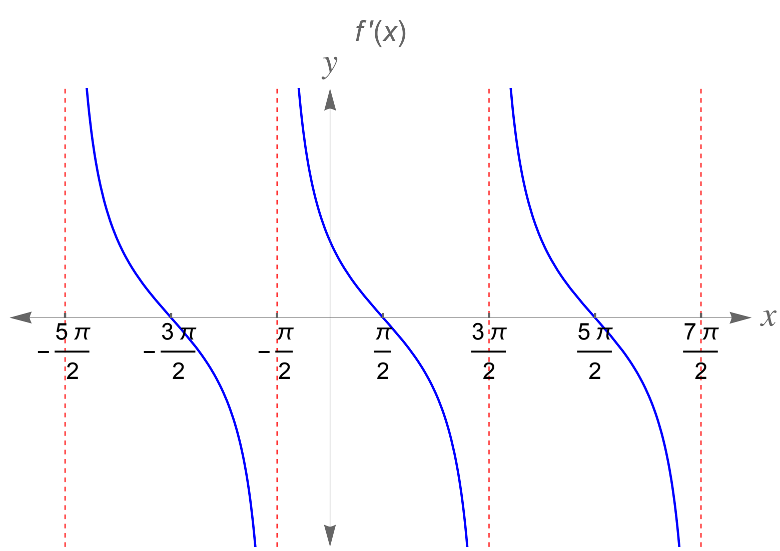
(a) $f''(x) = -\sec^2(x) < 0$ for all $x \in [0, \pi/2)$, thus f is concave down on the given domain.

(b) Note that
$$f'(x) = a + \frac{1}{\cos(x)}(-\sin(x)) = a - \tan(x),$$
which is defined for all $x \in [0, \pi/2)$. So the only critical points occur when $f'(x) = 0$, or
$$a - \tan(x) = 0 \implies a = \tan(x) \implies x = \arctan(a).$$
Now, since $f''(x) = -\sec^2(x) < 0$ for all $x \in [0, \pi/2)$, $f''(\arctan(a)) < 0$ meaning that f has a local maximum at $x = \arctan(a)$. Since this is the only critical point of f on the given domain, this must be the location of an *absolute* maximum. Thus $x = \arctan(a)$ is the x-coordinate of the absolute maximum of f on the given domain.

(c) (i) As noted in (b), $f'(x)$ exists on $[0, \pi/2)$. So, f is continuous on $[0, \pi/4]$ and differentiable on $(0, \pi/4)$. Thus, the Mean Value Theorem applies to $f(x)$ on $[0, \pi/4]$.

(ii) Note that $f(0) = 0 + \ln(\cos(0)) = \ln(1) = 0$ and
$$f(\pi/4) = \pi/4 + \ln(\cos(\pi/4)) = \frac{\pi}{4} + \ln\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} - \frac{1}{2}\ln(2).$$
Thus
$$\frac{f(\pi/4) - f(0)}{\pi/4 - 0} = \frac{\frac{\pi}{4} - \frac{1}{2}\ln(2) - 0}{\pi/4 - 0} = 1 - \frac{\frac{1}{2}\ln(2)}{\frac{\pi}{4}} = 1 - \frac{2\ln(2)}{\pi}.$$
Since $f'(x) = 1 - \tan(x)$ as above, we solve $f'(c) = 1 - \frac{2\ln(2)}{\pi}$, or
$$1 - \tan(c) = 1 - \frac{2\ln(2)}{\pi} \iff \tan(c) = \frac{2\ln(2)}{\pi} \iff c = \arctan\left(\frac{2\ln(2)}{\pi}\right).$$

4. (22 points) The graph of a **derivative** $f'(x)$ is shown in the following picture:



The dashed lines on the graph indicate vertical asymptotes of $y = f'(x)$. Please answer the following questions based on the given graph of $y = f'(x)$ given that both f and f' have a period of π and a domain of $\left(-\frac{5\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{7\pi}{2}\right)$. (No justifications are required for this problem.)

(a) On what open interval(s) (if any) is f increasing?

(b) For which x-coordinates (if any) does f have local maximums?

(c) For which x-coordinates (if any) does f have local minimums?

(d) On what open interval(s) (if any) is f concave up?

(e) For which x-coordinates (if any) does f have inflection point(s)?

(f) Use the grid below to sketch a graph of $f(x)$ for x-values between $-5\pi/2$ and $7\pi/2$, that has all of the features outlined in (a)-(e), as well as any asymptotes it should have. Use dashed lines to represent any asymptotes of $y = f(x)$.

Solution:

(a) f is increasing on $(-5\pi/2, -3\pi/2) \cup (-\pi/2, \pi/2) \cup (3\pi/2, 5\pi/2)$.

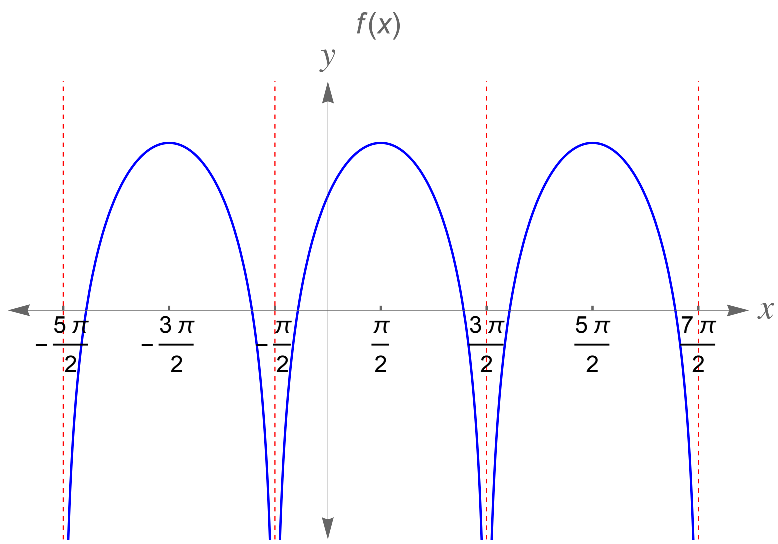
(b) f has local maxima at $x = -3\pi/2, \pi/2, 5\pi/2$ since f' switches from positive to negative there.

(c) f has no local minima.

(d) Since $f'(x)$ is itself decreasing on all the given intervals, $f''(x) < 0$ wherever it's defined, thus f is concave down on $(-5\pi/2, -\pi/2) \cup (-\pi/2, 3\pi/2) \cup (3\pi/2, 7\pi/2)$. This means that f is never concave up.

(e) f has no inflection points since f is concave down on its entire domain.

(f) A plausible graph of $f(x)$ could look like this:



5. (20 pts) Consider the function $g(x) = \dfrac{2\cosh(x)}{e^x}$ defined on $x \geq 0$.

(a) Show $g(x)$ is one-to-one for $x \geq 0$.

(b) Find the inverse function of $g(x)$. (Be sure to label your final answer as $g^{-1}(x)$.)

(c) Let $G(x) = \int_4^x g(t)\,dt$. Find $(G^{-1})'(0)$. (Hint: You do not need to find a formula for $G^{-1}(x)$ in order to complete this problem, and it will be easier if you do not attempt to do so.)

Solution:

(a) Recall, $\cosh(x) = \frac{e^x+e^{-x}}{2}$. So, $g(x) = 1 + e^{-2x}$, which implies $g'(x) = -2e^{-2x}$, which is negative for all x. Thus, g is always decreasing and g is one-to-one.

(b) We need to solve $x = 1 + e^{-2y}$ for y.

$$\begin{aligned} x &= 1 + e^{-2y} \\ \ln(x-1) &= -2y \\ y &= -\frac{1}{2}\ln(x-1) \\ g^{-1}(x) &= -\frac{1}{2}\ln(x-1). \end{aligned}$$

(c) Note that

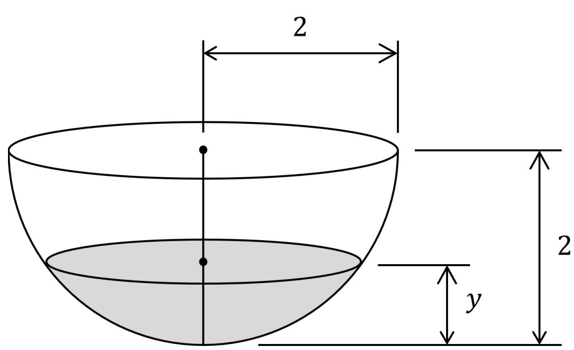
$$G(4) = \int_4^4 g(t)\,dt = 0,$$

so $G^{-1}(0) = 4$. Thus,

$$\begin{aligned} (G^{-1})'(0) &= \frac{1}{G'(G^{-1}(0))} \\ &= \frac{1}{G'(4)} \\ &= \frac{1}{1+e^{-8}}. \end{aligned}$$

6. (12 pts) A bowl is shaped as a hemisphere of radius 2 ft. When this bowl contains water having a depth of y ft, as depicted below, the corresponding volume of water in the bowl is given by the following function:

$$V = \pi y^2(2 - y/3) \qquad 0 \le y \le 2$$



Suppose this bowl is being filled in such a way that its water depth at time t minutes is

$$y(t) = \frac{t^2}{18} \quad \text{feet}, \quad 0 \le t \le 6.$$

How fast is the volume of the water increasing when the water is 1 ft deep? Simplify your answer fully and include the correct unit of measurement.

Solution:

The Chain Rule indicates that $\frac{dV}{dt} = \frac{dV}{dy} \cdot \frac{dy}{dt}$. Since $V = \pi y^2\left(2 - \frac{y}{3}\right) = 2\pi y^2 - \frac{\pi}{3}y^3$, then we have

$$\frac{dV}{dy} = 4\pi y - \pi y^2.$$

Note that $\left.\dfrac{dV}{dy}\right|_{y=1} = 4\pi - \pi = 3\pi.$

We are given $y(t) = \dfrac{1}{18}\,t^2$. So, $\dfrac{dy}{dt} = \dfrac{1}{9}\,t$. Since $y = 1$, we can solve for t:

$$1 = \frac{1}{18}\,t^2 \Rightarrow t = \sqrt{18} = 3\sqrt{2}.$$

So, we have

$$\left.\frac{dy}{dt}\right|_{y=1} = \left.\frac{dy}{dt}\right|_{t=3\sqrt{2}} = \frac{1}{9}\,(3\sqrt{2}) = \frac{\sqrt{2}}{3},$$

which implies

$$\frac{dV}{dt} = \frac{dV}{dy}\cdot\frac{dy}{dt} = 3\pi\cdot\frac{\sqrt{2}}{3} = \boxed{\sqrt{2}\,\pi\ \text{ft}^3\text{/min}}.$$

7. (14 pts) Atmospheric pressure, P (measured in millibars *mbar*), as a function of elevation above sea level x (measured in kilometers *km*), decreases according to the law of natural decay. (Note that sea level corresponds to $x = 0$ *km*.) In other words, the rate of change of atmospheric pressure with respect to elevation is proportional to the atmospheric pressure: $\frac{dP}{dx} = kP$ for some constant k.

   (a) Suppose that the pressure at sea level is 1000 mbar, and the pressure at the top of Mount Everest is 250 mbar, which is 9 km above sea level. Find a function which models atmospheric pressure.

   (b) At the top of Mount Integral, a newly discovered fictitious mountain in Colorado, the atmospheric pressure is 15% of the atmospheric pressure at sea level. How high is Mount Integral? Give an exact answer, and include the correct units.

   (c) Is Mount Integral higher or lower than Mount Everest? Just answer 'higher' or 'lower,' no justification is required for this part of this problem.

Solution:

(a) Since x measures the height above sea level, $P(0) = a = 1000$. At the top of Mount Everest, we have

$$250 = 1000e^{9k} \implies \frac{1}{4} = e^{9k} \implies -\frac{\ln(4)}{9} = k.$$

Hence, the atmospheric pressure is modeled by the function

$$P(x) = 1000e^{-\ln(4)x/9} = 1000(4)^{-x/9}.$$

(b) We need to solve $P(x) = 150$ for x.

$$\begin{aligned} 1000e^{-\ln(4)x/9} &= 150 \\ -\ln(4)x/9 &= \ln(3/20) \\ x &= \frac{9\ln(20/3)}{\ln 4}\ \textit{mbar}. \end{aligned}$$

(c) Higher. This is apparent because the pressure decreases with higher elevations.