Final Exam
Fall 2023

| Name |  |
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| Instructor | Lecture Section |

This exam is worth 150 points and has 7 problems.
Make sure all of your work is written in the blank spaces provided. If your solutions do not fit, there is additional space at the end of the test. Be sure to make a note indicating the page number where the work is continued or it will not be graded.
Show all work and simplify your answers. Name any theorem that you use. Answers with no justification will receive no points unless the problem explicitly states otherwise.
Notes, papers, calculators, cell phones, and other electronic devices are not permitted.

## End of Exam Check List

1. If you finish the exam before $9: 45 \mathrm{AM}$ :

- Go to the designated area to scan and upload your exam to Gradescope.
- Verify that your exam has been correctly uploaded and all problems have been labeled.
- Leave the physical copy of the exam with your proctors.

2. If you finish the exam after 9:45 AM:

- Please wait in your seat until 10:00 AM.
- When instructed to do so, scan and upload your exam to Gradescope at your seat.
- Verify that your exam has been correctly uploaded and all problems have been labeled.
- Leave the physical copy of the exam with your proctors.


## Formulas

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\begin{gathered}
\sin (2 \theta)=2 \sin \theta \cos \theta \quad \cos (2 \theta)=2 \cos ^{2} \theta-1=1-2 \sin ^{2} \theta \\
\sum_{i=1}^{n} i=\frac{n(n+1)}{2} \quad \sum_{i=1}^{n} i^{2}=\frac{n(n+1)(2 n+1)}{6} \quad \sum_{i=1}^{n} i^{3}=\left(\frac{n(n+1)}{2}\right)^{2} \\
\int \frac{1}{\sqrt{1-x^{2}}} d x=\arcsin (x)+C \quad \int \frac{1}{1+x^{2}} d x=\arctan (x)+C
\end{gathered}
$$

1. (40 pts) The following problems are unrelated.
(a) Find the equation of the tangent line of $y=\ln \left(x^{2} \ln x\right)$ at $x=e$.
(b) Evaluate $\int_{-\frac{\ln (2)}{3}}^{\frac{-\ln (2 / \sqrt{3})}{3}} \frac{1}{e^{-3 x} \sqrt{1-e^{6 x}}} d x$. (Just to be sure it's clear, the lower limit of integration is $-\frac{\ln (2)}{3}$ and the upper limit is $-\frac{\ln (2 / \sqrt{3})}{3}$.)
(c) Evaluate $\lim _{x \rightarrow \infty} \frac{\log _{4}(x+6)}{\log _{3}(x)}$.
(d) Evaluate $\sin ^{-1}(\sin (3 \pi / 4))-\tan ^{-1}(\tan (4 \pi / 5))$. Fully simplify your answer.
2. (16 pts) Let $f(x)=\left\{\begin{array}{cc}x^{2}-x & x \geq 0 \\ |x| & x<0\end{array}\right.$
(a) Find $f^{\prime}(x)$. Your answer will also be a piecewise-defined function. Specifically, do the following:

- Use any tools from our class to find $f^{\prime}(x)$ when $x \neq 0$.
- Use the definition of the derivative to find $f^{\prime}(0)$ or to argue that $f^{\prime}(0)$ does not exist.
(b) Use your answer from (a) and the definition of the derivative to find $f^{\prime \prime}(0)$, or to argue that it does not exist.
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3. (26 pts) Consider the function $f(x)=a x+\ln (\cos (x))$ on the domain $[0, \pi / 2)$, where $a>0$ is a constant.
(a) Is $f(x)$ concave up anywhere on this domain? If so, give the interval(s) where that occurs.
(b) Find the $x$-coordinate of the absolute maximum of $f(x)$ on the given domain. Note: your answer will be in terms of $a$. Be sure to justify that you have located the absolute maximum.
(c) Now, set $a=1$.
(i) Determine if the Mean Value Theorem applies to $f(x)$ on $[0, \pi / 4]$. (Clearly note if the hypotheses of the theorem apply or do not apply here.)
(ii) If the answer to (i) is yes, find the value of $c$ that satisfies the conclusion of the Mean Value Theorem.
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4. (22 points) The graph of a derivative $f^{\prime}(x)$ is shown in the following picture:


The dashed lines on the graph indicate vertical asymptotes of $y=f^{\prime}(x)$. Please answer the following questions based on the given graph of $y=f^{\prime}(x)$ given that both $f$ and $f^{\prime}$ have a period of $\pi$ and a domain of $\left(-\frac{5 \pi}{2},-\frac{\pi}{2}\right) \cup$ $\left(-\frac{\pi}{2}, \frac{3 \pi}{2}\right) \cup\left(\frac{3 \pi}{2}, \frac{7 \pi}{2}\right)$. (No justifications are required for this problem.)
(a) On what open interval(s) (if any) is $f$ increasing?
(b) For which $x$-coordinates (if any) does $f$ have local maximums?
(c) For which $x$-coordinates (if any) does $f$ have local minimums?
(d) On what open interval(s) (if any) is $f$ concave up?
(e) For which $x$-coordinates (if any) does $f$ have inflection point(s)?
(f) Use the grid below to sketch a graph of $f(x)$ for $x$-values between $-5 \pi / 2$ and $7 \pi / 2$, that has all of the features outlined in (a)-(e), as well as any asymptotes it should have. Use dashed lines to represent any asymptotes of $y=f(x)$.

5. (20 pts) Consider the function $g(x)=\frac{2 \cosh (x)}{e^{x}}$ defined on $x \geq 0$.
(a) Show $g(x)$ is one-to-one for $x \geq 0$.
(b) Find the inverse function of $g(x)$. (Be sure to label your final answer as $g^{-1}(x)$.)
(c) Let $G(x)=\int_{4}^{x} g(t) d t$. Find $\left(G^{-1}\right)^{\prime}(0)$. (Hint: You do not need to find a formula for $G^{-1}(x)$ in order to complete this problem, and it will be easier if you do not attempt to do so.)
6. (12 pts) A bowl is shaped as a hemisphere of radius 2 ft . When this bowl contains water having a depth of $y \mathrm{ft}$, as depicted below, the corresponding volume of water in the bowl is given by the following function:

$$
V=\pi y^{2}(2-y / 3) \quad 0 \leq y \leq 2
$$



Suppose this bowl is being filled in such a way that its water depth at time $t$ minutes is

$$
y(t)=\frac{t^{2}}{18} \quad \text { feet, } \quad 0 \leq t \leq 6
$$

How fast is the volume of the water increasing when the water is 1 ft deep? Simplify your answer fully and include the correct unit of measurement.
7. (14 pts) Atmospheric pressure, $P$ (measured in millibars mbar), as a function of elevation above sea level $x$ (measured in kilometers km ), decreases according to the law of natural decay. (Note that sea level corresponds to $x=0$ km .) In other words, the rate of change of atmospheric pressure with respect to elevation is proportional to the atmospheric pressure: $\frac{d P}{d x}=k P$ for some constant $k$.
(a) Suppose that the pressure at sea level is 1000 mbar , and the pressure at the top of Mount Everest is 250 mbar , which is 9 km above sea level. Find a function which models atmospheric pressure.
(b) At the top of Mount Integral, a newly discovered fictitious mountain in Colorado, the atmospheric pressure is $15 \%$ of the atmospheric pressure at sea level. How high is Mount Integral? Give an exact answer, and include the correct units.
(c) Is Mount Integral higher or lower than Mount Everest? Just answer 'higher' or 'lower,' no justification is required for this part of this problem.

