1. (24 pts) The following problems are unrelated. For each, be sure to fully simplify your answers.

   (a) Evaluate $\displaystyle\int_0^2 \frac{x+1}{\sqrt{x^2+2x+4}}\,dx$.

   (b) Evaluate $\displaystyle\frac{d}{dx}\left(\int_x^{x^2} \sqrt{t}\cos(t)\,dt\right)$ where $x > 0$.

   (c) Suppose $\displaystyle\int_1^{-3} (2x+h(x))\;dx = 10$ and $\displaystyle\int_0^1 5h(x)\,dx = -8$. Find the average value of $h(x)$ over $[-3,0]$.

Solution:

(a) We proceed by substitution. Let $u = x^2+2x+4$. Then $du = (2x+2)dx$ and so $(x+1)dx = \dfrac{du}{2}$. We also see that the new limits of integration are an upper limit of $u = 2^2+2(2)+4 = 12$ and a lower limit of $u = 0^2+2(0)+4 = 4$. Then, we have

$$\begin{aligned}\int_0^2 \frac{x+1}{\sqrt{x^2+2x+4}}dx &= \frac{1}{2}\int_4^{12}\frac{1}{\sqrt{u}}du\\ &= \frac{1}{2}\left[2u^{\frac{1}{2}}\right]_4^{12}\\ &= 12^{\frac{1}{2}} - 4^{\frac{1}{2}}\\ &= 2(\sqrt{3}-1).\end{aligned}$$

(b) Here we must use the Fundamental Theorem of Calculus and the Chain Rule:

$$\begin{aligned}\frac{d}{dx}\int_x^{x^2}\sqrt{t}\cos(t)dt &= \sqrt{x^2}\cos(x^2)\frac{d}{dx}\left(x^2\right) - \sqrt{x}\cos(x)\\ &= 2x^2\cos(x^2) - \sqrt{x}\cos(x).\end{aligned}$$

(c) The first provided condition can be manipulated:

$$\begin{aligned}\int_1^{-3}(2x+h(x))\;dx &= 10\\ \int_{-3}^{1}(2x+h(x))\;dx &= -10\\ \int_{-3}^{1}2x\,dx + \int_{-3}^{1}h(x)\,dx &= -10\\ -8 + \int_{-3}^{1}h(x)\,dx &= -10\\ \int_{-3}^{1}h(x)\,dx &= -2.\end{aligned}$$

Likewise, we can manipulate the second given condition:

$$\int_0^1 5h(x)\,dx = -8$$
$$\int_0^1 h(x)\,dx = -\frac{8}{5}.$$

So, we have

$$\int_{-3}^0 h(x)\,dx = -2 - \left(-\frac{8}{5}\right) = -\frac{2}{5}$$

which means the average value is

$$\frac{1}{0-(-3)} \int_{-3}^0 h(x)\,dx = -\frac{2}{15}.$$

2. (20 pts) The following problems are unrelated. For each, be sure to fully simplify your answers.

   (a) An object moves along an axis starting at time $t = 0$. It moves with acceleration $\sin(t)$ meters per second squared with an initial velocity of 3 meters per second and an initial position of 2 meters. Find the position of this object at $t = \frac{3\pi}{2}$ seconds. (Provide an exact answer in terms of π. Do not round or approximate your final answer.)

   (b) Consider the integral $\displaystyle\int_0^2 x^3\,dx$. In this problem, you will evaluate this integral in two different ways.

      i. Evaluate this integral using the Fundamental Theorem of Calculus Part 2 (also known as the Evaluation Theorem).

      ii. Separately, write the integral as the limit of a right-hand Riemann sum using n rectangles of equal width. Evaluate this sum and limit without using your work from (i).

Solution:

(a) We have $a(t) = \sin(t)$. We then antidifferentiate to get $v(t) = -\cos(t) + C_1$. Using $v(0) = 3$ we get $C_1 = 4$, which means

$$v(t) = -\cos(t) + 4.$$

We now antidifferentiate this to get $s(t) = -\sin(t) + 4t + C_2$. Using $s(0) = 2$ we get $C_1 = 4$, which means

$$s(t) = -\sin(t) + 4t + 2,$$

which yields

$$s\left(3\pi/2\right) = -(-1) + 6\pi + 2 = 3 + 6\pi \text{ meters.}$$

(b) i.

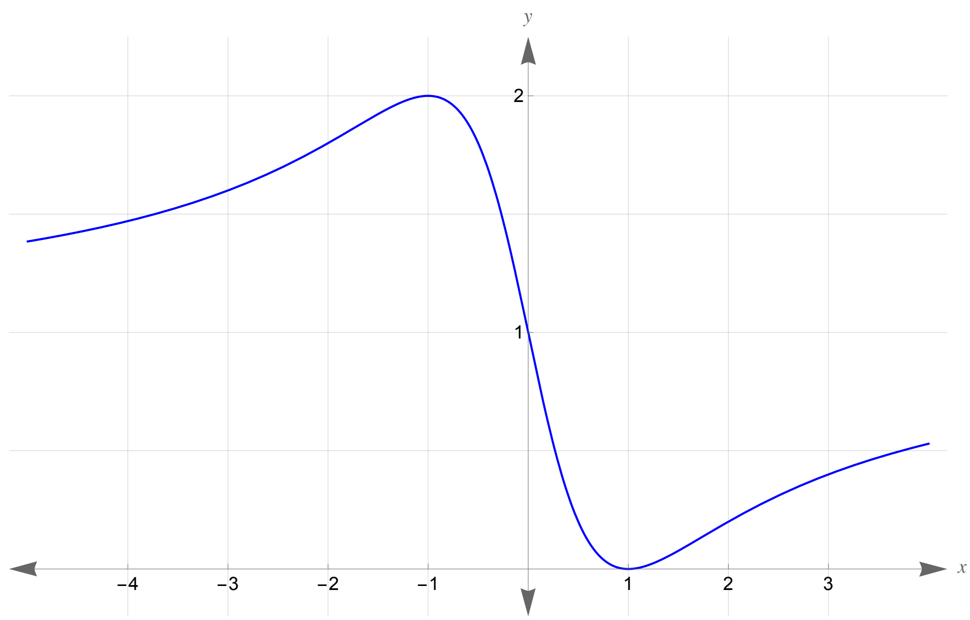
$$\int_0^2 x^3\,dx = \frac{1}{4}\left(2^4 - 0^4\right) = 4.$$

ii. Note that $\triangle x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$, and that the right endpoints of the subintervals will be given by $x_i = 0 + i\triangle x = \frac{2i}{n}$. So, we have

$$\int_0^2 x^3\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^3 \left(\frac{2}{n}\right)$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{16}{n^4} \sum_{i=1}^{n} i^3 \\
&= \lim_{n\to\infty} \frac{16}{n^4} \cdot \frac{n^2(n+1)^2}{4} \\
&= \lim_{n\to\infty} \left(4 + \frac{8}{n} + \frac{4}{n^2}\right) \\
&= 4
\end{aligned}
$$

3. (10 pts) The following is a graph of $y = f(x)$. Be sure to scan and submit this graph as part of your answer because you will draw on it in addition to answering a few questions (below).



(a) What is the x-value of the only zero of f? (Note: The only zero of f is shown in the above graph.)

(b) With $x_1 = -2$, after one iteration of Newton's Method will x_2 be closer to or farther away from the zero in part (a)?

i. Answer '**closer**' or '**farther**.'

ii. On the provided graph draw the relevant line supporting your answer from (i). *Label this line as (b).*

(c) Give an x-value x_1 for which Newton's method definitely will **not** converge.

(d) With $x_1 = 3$, draw a line on the provided graph illustrating one step of Newton's Method. *Label this line as (d).*

(e) What will $\lim\limits_{n\to\infty} x_n$ equal when $x_1 = 3$? (That is, with an initial guess of $x_1 = 3$ and repeated application of Newton's Method, to what value will the x_n's approach?)

Solution:

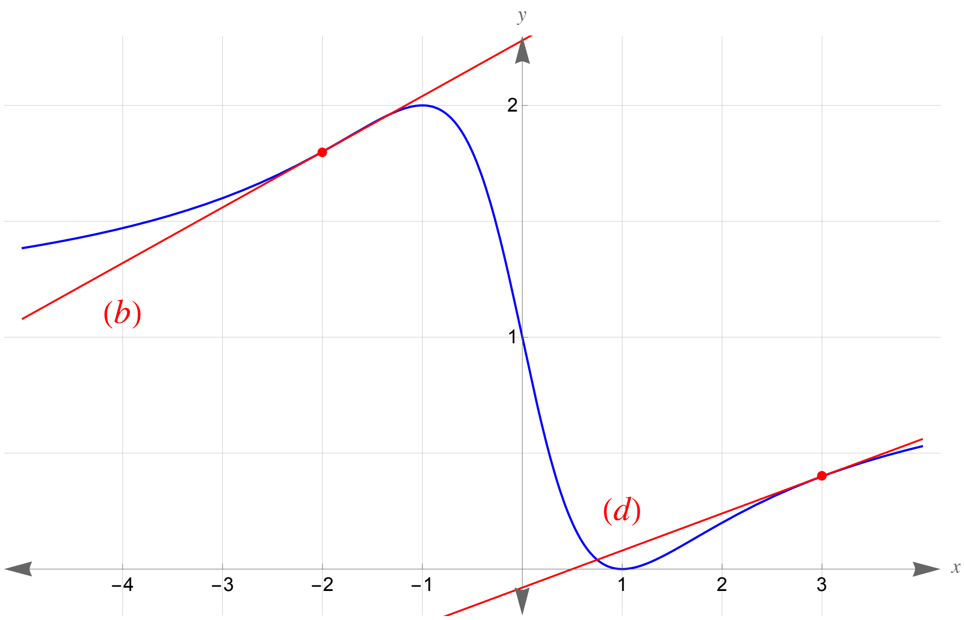
(a) $x = 1$.

(b) **Farther.** The tangent line to the graph at $x = -2$ will intersect the x-axis far to the left, farther from $x = 1$ than the starting seed value of -2.

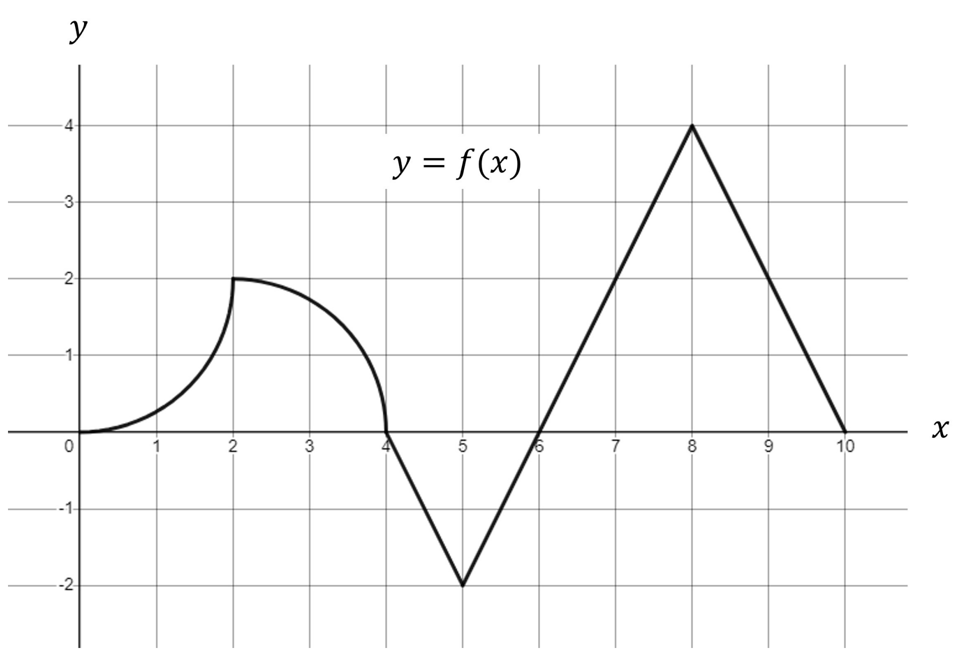
(c) $x_1 = -1$. The tangent line there is horizontal, meaning Newton's method will fail with this value for x_1.

(d) See the line on the graph below that is tangent to the curve at $x = 3$.

(e) It appears that the algorithm will converge to $x = 1$ with $x_1 = 3$, meaning that $\lim_{n\to\infty} x_n = 1$.



4. (22 pts) Consider the function $f(x)$ defined over $[0, 10]$ that is graphed below. The graph consists of three line segments and two quarter circles.



(a) Approximate $\int_0^{10} f(x)dx$ by using five rectangles of equal width with the righthand endpoint rule.

(b) Determine the exact value of $\int_0^{10} f(x)dx$.

(c) Let $g(x) = \int_0^x f(t)dt$.

i. Identify where the function $g(x)$ is concave up. Express your answer using interval notation.

ii. Given that $g(6) = 2$, find an equation of the line that is tangent to the curve $y = g(x)$ at $x = 7$.

Solution:

(a)

$$\int_0^{10} f(x)dx \approx R_5 = \Delta x[f(2) + f(4) + f(6) + f(8) + f(10)]$$
$$= (2)[2 + 0 + 0 + 4 + 0] = \boxed{12}$$

(b) Using geometry (areas of quarter circles and triangles) and remembering that area below the x-axis is negative, we have the following:

$$\int_0^{10} f(x)dx = \int_0^{2} f(x)dx + \int_2^{4} f(x)dx + \int_4^{6} f(x)dx + \int_6^{10} f(x)dx$$
$$= \left[(2)(2) - \frac{\pi(2)^2}{4}\right] + \left[\frac{\pi(2)^2}{4}\right] - \frac{1}{2}(2)(2) + \frac{1}{2}(4)(4)$$
$$= 4 - 2 + 8 = \boxed{10}$$

(c) $g(x) = \displaystyle\int_0^{x} f(t)dt.$

i. The function $g(x)$ is concave up where $g''(x) = f'(x) > 0$, which occurs on the intervals over which $f(x)$ is increasing. The graph indicates that this occurs on the intervals $\boxed{(0, 2) \cup (5, 8)}$

ii. The slope of the line that is tangent to the curve $y = g(x)$ at $x = 7$ is given by $g'(7) = f(7) = 2$.

To find the value of $g(7)$, use part 2 of the Fundamental Theorem of Calculus (the Evaluation Theorem).

$$\int_6^{7} f(x)dx = g(7) - g(6)$$

We are given that $g(6) = 2$. The figure indicates that $\displaystyle\int_6^{7} f(x)dx$ is the area of a right triangle with a base of 1 and a height of 2, which equals $\frac{1}{2}(1)(2) = 1$. It follows that $1 = g(7) - 2$, which implies that $g(7) = 3$.

Therefore, an equation of the tangent line is $\boxed{y - 3 = 2(x - 7)}$

5. (12 pts) Using the grid below, sketch the graph of a **single function**, $y = f(x)$ with each of the following characteristics. (Sketch dashed lines to indicate any asymptotes that are present. The concavity of your graph should be clear.)

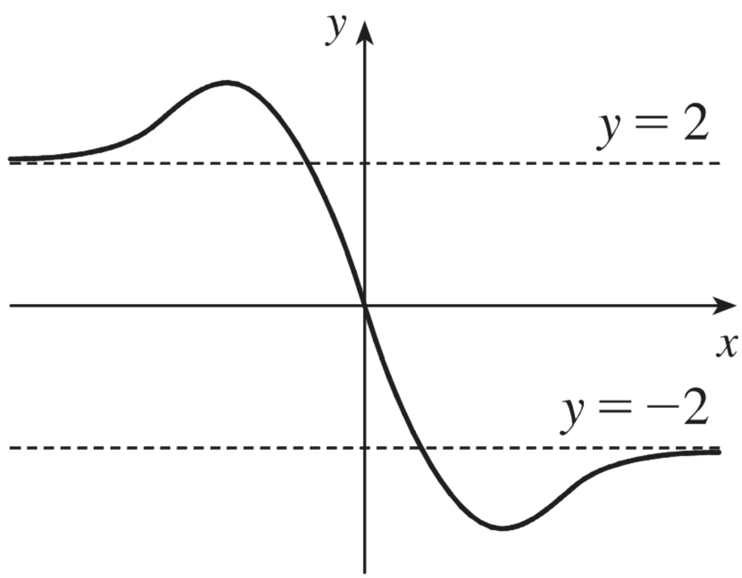
$$f(-x) = -f(x) \text{ for all } x, \quad f'(x) < 0 \text{ for } 0 < x < 3$$

$$f'(3) = 0, \quad f'(x) > 0 \text{ for } x > 3$$

$$f''(x) < 0 \text{ for } x > 4, \quad f''(x) > 0 \text{ for } 0 < x < 4$$

$$\lim_{x\to-\infty} f(x) = 2, \quad f \text{ is continuous for all } x$$

Solution: f is odd and thus from the given $\lim_{x\to\infty} f(x) = -2$. f has a local minimum at $x = 3$ and an inflection point at $x = 4$ where it switches from concave up to down. A plausible graph would look like this:



6. (12 pts) A box with an open top is to be constructed from a square piece of cardboard, 6 ft by 6 ft, by cutting out identical squares from each of the four corners and bending up the sides.

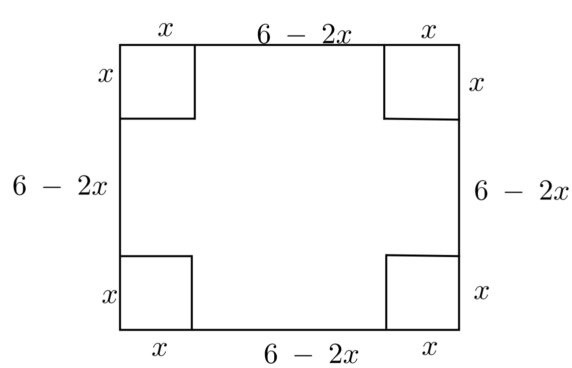
   (a) Draw a diagram of the flat cardboard sheet with squares cut out of the corners. Make sure you label the length of any parts which are cut out, and also the lengths of the remaining sides.

   (b) What are the dimensions of the square you need to cut out of each corner so that your box has maximum volume? Include the correct units of measurement in your final answer. (Be sure you have justified that you have found the absolute maximum volume.)

   (c) What is the maximum volume of the box? Include the correct units of measurement in your final answer.

   Solution:

(a)



(b) The volume is given by the expression

$$V(x) = x(6-2x)^2 = 4x^3 - 24x^2 + 36x$$

where x is the side length of the square cut out from each corner. Hence, the domain for $V(x)$ is $0 \leq x \leq 3$, but $V(0) = 0$ and $V(3) = 0$. The derivative is given by

$$V'(x) = 12x^2 - 48x + 36,$$

and setting it equal to zero simplifying gives the relation

$$0 = x^2 - 4x + 3 = (x-3)(x-1).$$

Our critical points are thus $x = 1$ and $x = 3$, but $x = 3$ is an endpoint and we already found that $V(3) = 0$. Since $V(1) = 16$ is greater than the volume of 0 at the endpoints of $x = 0$ and $x = 3$, cutting out a 1 ft by 1 ft from each corner gives a box of maximum volume.

(c) We calculated in the work above that $V(1) = 16$ cubic feet is the maximum volume for a box constructed in this way.