1. (27 points) The following two problems are not related.

   (a) Suppose that A and B are constants. Find the derivative of $h(x) = \dfrac{A\sin(x)}{x} + \tan(Bx)$. (Please do not simplify your final answer.)

   (b) Consider $s(x) = \sqrt{x^2+1} \cdot \cos x$.

   i. Find the value of $s'\left(\dfrac{\pi}{2}\right)$.

   ii. Find the formula for the linearization of $y = s(x)$ at $a = \dfrac{\pi}{2}$. (Please do not simplify your final answer. Your final answer will be in terms of π.)

   iii. Use your linearization from (ii) to approximate $s\left(\frac{3}{2}\right)$. (Please do not simplify your final answer. Your final answer will be in terms of π.)

Solution:

(a) Using the quotient rule on the first term, and the chain rule on the second term, gives

$$\begin{aligned} h'(x) &= \frac{x(A\cos x) - (A\sin x)(1)}{x^2} + B\sec^2(Bx) \\ &= \frac{A(x\cos x - \sin x)}{x^2} + B\sec^2(Bx) \end{aligned}$$

(b) i. Taking the derivative, we find that

$$\begin{aligned} s'(x) &= \left(\sqrt{x^2+1}\right)' \cos x + \left(\sqrt{x^2+1}\right)(\cos x)' \\ &= \left(2x \cdot \frac{1}{2}(x^2+1)^{-1/2}\right)\cos x + \sqrt{x^2+1}(-\sin x) \\ &= \frac{x\cos x}{\sqrt{x^2+1}} - \sqrt{x^2+1} \cdot \sin x. \end{aligned}$$

Evaluating the derivative at $x = \pi/2$ yields

$$\begin{aligned} s'\left(\frac{\pi}{2}\right) &= \frac{(\pi/2)\cos(\pi/2)}{\sqrt{(\pi/2)^2+1}} - \sqrt{(\pi/2)^2+1} \cdot \sin(\pi/2) \\ &= -\frac{\sqrt{\pi^2+4}}{2} \end{aligned}$$

ii. The linearization (tangent line) is given by

$$\begin{aligned} L(x) &= s\left(\frac{\pi}{2}\right) + s'\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right) \\ &= -\frac{\sqrt{\pi^2+4}}{2}\left(x - \frac{\pi}{2}\right). \end{aligned}$$

iii. We have that $s(x) \approx L(x)$ for x “near” $\frac{\pi}{2}$. So, we have

$$\begin{aligned} s\left(\frac{3}{2}\right) &\approx L\left(\frac{3}{2}\right) \\ s\left(\frac{3}{2}\right) &\approx -\frac{\sqrt{\pi^2+4}}{2}\left(\frac{3-\pi}{2}\right). \end{aligned}$$

2. (16 points) Consider the curve defined by $8x + 2xy + y^3 = 11$. Complete the following.

   (a) Find y' at the point $(1, 1)$.

   (b) Find y'' at the point $(1, 1)$.

Solution:

(a) We start by using implicit differentiation on the equation of the curve:

$$8 + 2(xy' + y) + 3y^2y' = 0$$

Next, we substitute $x = 1$ and $y = 1$, and then solve for y':

$$\begin{aligned} 8 + 2(y' + 1) + 3y' &= 0 \\ 5y' + 10 &= 0 \\ y' &= -2. \end{aligned}$$

(b) We must use implicit differentiation once more to get an equation with y''.

$$\begin{aligned} \frac{d}{dx}\left(8 + 2(xy' + y) + 3y^2y'\right) &= \frac{d}{dx}(0) \\ 2(xy'' + y' + y') + 3(y^2y'' + 2y(y')^2) &= 0 \end{aligned}$$

Next, we substitute $x = 1$, $y = 1$, and $y' = -2$ and then solve for y'':

$$\begin{aligned} 2(y'' + 2(-2)) + 3(y'' + 2(-2)^2) &= 0 \\ y'' &= -\frac{16}{5}. \end{aligned}$$

3. (18 pts) Given $f(x) = x^{4/3} + 4x^{1/3} + 4x^{-2/3}$

   (a) Determine the x-coordinate(s) for all critical number(s) of f.

   (b) Determine the interval(s) where f is decreasing.

   (c) Determine the x-coordinate(s) of all local maxima and minima of f. (Clearly indicate which x-coordinates correspond to a local maximum and which correspond to a local minimum.)

Solution:

(a) Note that

$$f'(x) = \frac{4}{3}x^{-5/3}(x^2 + x - 2) = \frac{4(x-1)(x+2)}{3x^{5/3}}$$

which is zero when $x = -2, 1$ and undefined when $x = 0$. $x = 0$ is not in the domain of f, so the only critical points occur at $x = -2, 1$.

(b) We see that $f'(x) < 0$ when x is in $(-\infty, -2) \cup (0, 1)$ and $f'(x) > 0$ when x is in $(-2, 0) \cup (1, \infty)$. So, f is decreasing from $(-\infty, -2)$ and $(0, 1)$, increasing from $(-2, 0)$ and $(1, \infty)$. (We still have to consider the sign of f' on both sides of $x = 0$).

(c) Applying the first derivative test at the two critical points, we see that f has a local minimum at $x = -2$ and at $x = 1$.

4. (13 points) Consider $g(x) = 3\sqrt{4-x} + 5$. Use the definition of the derivative to show that $g'(x) = \dfrac{-3}{2\sqrt{4-x}}$.
(Note: You must use the definition of the derivative to earn any credit on this problem.)

Solution:

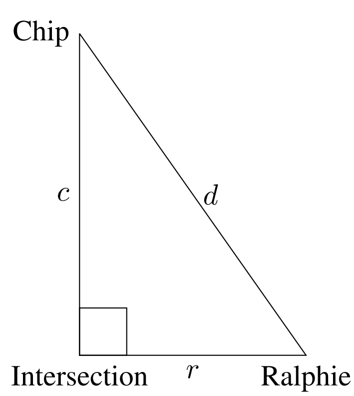
$$\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h\to 0} \frac{\left(3\sqrt{4-(x+h)} + 5\right) - \left(3\sqrt{4-x} + 5\right)}{h} \\
&= 3\lim_{h\to 0} \frac{\sqrt{4-x-h} - \sqrt{4-x}}{h} \cdot \frac{\sqrt{4-x-h} + \sqrt{4-x}}{\sqrt{4-x-h} + \sqrt{4-x}} \\
&= 3\lim_{h\to 0} \frac{(4-x-h) - (4-x)}{h(\sqrt{4-x-h} + \sqrt{4-x})} \\
&= 3\lim_{h\to 0} \frac{-h}{h(\sqrt{4-x-h} + \sqrt{4-x})} \\
&= 3\lim_{h\to 0} \frac{-1}{\sqrt{4-x-h} + \sqrt{4-x}} \\
&= \frac{-3}{2\sqrt{4-x}}
\end{aligned}$$

5. (13 points) Ralphie is riding his bicycle east away from an intersection at 12 kilometers per hour when he is 2 kilometers east of the intersection. Chip is also riding a bicycle, but he is heading south towards that same intersection at a speed of 17 kilometers per hour when he is 3 kilometers north of the intersection. What is the rate of change of the straight-line distance between Ralphie and Chip at that moment? Include the correct unit of measurement in your answer.

Solution:

First, we label a right triangle that represents the situation. We have

$c =$ distance between Chip and the intersection,
$r =$ distance between the intersection and Ralphie, and
$d =$ distance between Chip and Ralphie.



In order to solve the problem, we want to find d' when

$$\begin{aligned} c &= \text{ 3 kilometers,} \\ c' &= \text{ -17 kilometers per hour,} \\ r &= \text{ 2 kilometers, and} \\ r' &= \text{ 12 kilometers per hour.} \end{aligned}$$

Note that the variables c, r, and d are related via the Pythagorean Theorem:

$$c^2 + r^2 = d^2.$$

If we differentiate and then solve for d', we have

$$d' = \frac{cc' + rr'}{d}.$$

We can now plug in the values of our variables and rates (derivatives) at that moment. Note that we can use the Pythagorean theorem to determine that

$$d = \sqrt{c^2 + r^2} = \sqrt{13} \text{ kilometers}$$

at that moment. Thus, we have that the rate of change between Chip and Ralphie at that moment is

$$d' = \frac{3(-17) + 2(12)}{\sqrt{13}} = -\frac{27}{\sqrt{13}} \text{ kilometers per hour.}$$

6. (13 points) Suppose that $r(0) = 2$ and $r'(x) \leq 6$ for all values of x. How large can $r(3)$ possibly be? Correctly use a theorem to justify your answer. (You should state the name of the theorem used and clearly show that its hypotheses are satisfied.)

   Solution:

   Since f is given as differentiable for all x, it is also continuous everywhere. So, f is continuous on $[0, 3]$ and differentiable on $(0, 3)$, meaning the Mean Value Theorem applies to f on $[0, 3]$ So, there is an x-value c for which $0 < c < 3$ and

$$\frac{r(3) - r(0)}{3 - 0} = r'(c) \implies r(3) - r(0) = 3r'(c) \implies r(3) = r(0) + 3r'(c) = 2 + 3r'(c).$$

   But $r'(c) \leq 6$ by the given, thus $r(3) \leq 2 + 3(6) = 20$.