1. (20 points) Problems (a) and (b) are not related.

   (a) Determine all solutions of $\sin^2(x) = \sin(2x) - \sin(x)\cos(x)$ in $[0, 2\pi)$.

   (b) If θ is in $[\pi/2, \pi]$ and $\tan(\theta) = -5/8$, evaluate

      i. $\sin(\theta)$

      ii. $\cos(2\theta)$

   Solution:

   (a) $\sin^2(x) = \sin(2x) - \sin(x)\cos(x)$
   $\sin^2(x) = 2\sin(x)\cos(x) - \sin(x)\cos(x)$
   $\sin^2(x) = \sin(x)\cos(x)$
   $\sin^2(x) - \sin(x)\cos(x) = 0$
   $\sin(x)(\sin(x) - \cos(x)) = 0$
   So either $\sin(x) = 0$ or $(\sin(x) - \cos(x)) = 0$
   $x = 0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}$

   (b) We can consider the right triangle in the second quadrant with leg lengths $x = -8$ and $y = 5$:
   $r^2 = (-8)^2 + (5)^2 = 64 + 25 = 89$
   $r = \sqrt{89}$

      i. $\sin(\theta) = \frac{y}{r} = \frac{5}{\sqrt{89}}$.

      ii. $\cos(2\theta) = 1 - 2\sin^2\theta = 1 - \frac{50}{89} = \frac{39}{89}$

2. (24 points) Evaluate the following limits and simplify your answers. If a limit does not exist, clearly state this. If you use a theorem, clearly state its name and show that its hypotheses are satisfied.
(Reminder: You may not use L'Hopital's Rule or Dominance of Powers in any solutions on this exam.)

   (a) $\displaystyle\lim_{x\to-2} \frac{|x^2+5x+6|}{x+2}$

   (b) If $4x^3 - 6x^2 \le f(x) \le x^4 - 4x + 1$ for all x, evaluate $\displaystyle\lim_{x\to 1} f(x)$.

   (c) $\displaystyle\lim_{y\to\pi/4} \left(\frac{1-\tan y}{\sin y - \cos y}\right)$

   Solution:

   (a) This limit is an $\frac{0}{0}$-indeterminate form, so we do not immediately know its value. Next, we note that $\dfrac{|x^2+5x+6|}{x+2} = \dfrac{|x+2|}{x+2}|x+3|$. Note that $\frac{|x+2|}{x+2} = 1$ for $x > -2$ and $\frac{|x+2|}{x+2} = -1$ for $x < -2$. We will use this observation and consider the one-sided limits as x approaches -2.

$$\begin{aligned}\lim_{x\to-2^+} \frac{|x^2+5x+6|}{x+2} &= \lim_{x\to-2^+} \frac{|x+2|}{x+2}|x+3| \\ &= \lim_{x\to-2^+} |x+3|\end{aligned}$$

$$= |-2+3|$$
$$= 1$$

$$\begin{aligned}\lim_{x\to -2^-} \frac{|x^2+5x+6|}{x+2} &= \lim_{x\to -2^-} \frac{|x+2|}{x+2}|x+3| \\ &= \lim_{x\to -2^-} -|x+3| \\ &= -|-2+3| \\ &= -1\end{aligned}$$

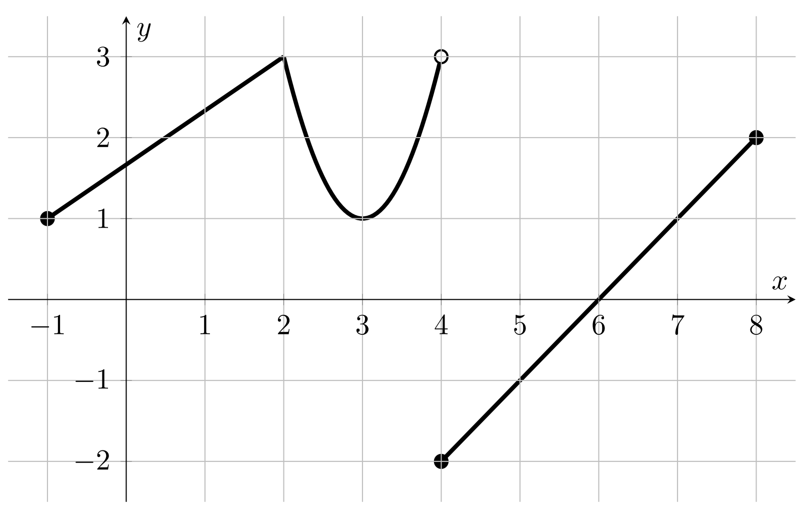
Since the two one-sided limits disagree, then we have shown $\lim_{x\to -2} \frac{|x^2+5x+6|}{x+2}$ does not exist.

(b) Since $\lim_{x\to 1} 4x^3 - 6x^2 = -2$ and $\lim_{x\to 1} x^4 - 4x + 1 = -2$, and since $f(x)$ is between these two functions for all x (as given), by the Squeeze Theorem $\lim_{x\to 1} f(x) = -2$ as well.

(c) This limit is an $\frac{0}{0}$-indeterminate form, so we do not immediately know its value.

$$\begin{aligned}\lim_{y\to\pi/4}\left(\frac{1-\tan y}{\sin y - \cos y}\right) &= \lim_{y\to\pi/4}\left(\frac{1-\frac{\sin y}{\cos y}}{\sin y - \cos y}\right)\cdot\frac{\cos y}{\cos y} \\ &= \lim_{y\to\pi/4}\left(\frac{\cos y - \sin y}{(\sin y - \cos y)\cos y}\right) \\ &= \lim_{y\to\pi/4}\frac{-1}{\cos y} \\ &= \frac{-1}{\frac{\sqrt{2}}{2}} \\ &= -\sqrt{2}.\end{aligned}$$

3. (12 points) The function $s(x)$ is graphed in its entirety below. It consists of two line segments and a portion of a parabola. Use the graph to answer the questions below. For this problem, no justification is required for your final answers.



(a) What is the domain of $r(x) = \frac{1}{s(x)}$? State your answer using interval notation.

(b) On what interval(s) is $r(x) = \dfrac{1}{s(x)}$ continuous? State your answer using interval notation.

(c) What is the domain of $v(x) = \sqrt{s(x)}$? State your answer using interval notation.

(d) Provide a complete formula for $s(x)$ as a piecewise-defined function.

Solution:

(a) $[-1, 6) \cup (6, 8]$

(b) $[-1, 4) \cup (4, 6) \cup (6, 8]$

(c) $[-1, 4) \cup [6, 8]$

(d)

$$s(x) = \begin{cases} \frac{2}{3}(x+1) + 1 & , \ -1 \le x \le 2 \\ 2(x-3)^2 + 1 & , \ 2 < x < 4 \\ x - 6 & , \ 4 \le x \le 8 \end{cases}$$

4. (18 points) Consider the function $g(x) = \dfrac{\sin x}{x(x - \pi/2)}$.

(a) Identify all values of x, if any, for which $g(x)$ has a vertical asymptote. Justify your answer by evaluating the appropriate limit(s).

(b) For what value(s) of a is the following piecewise function $h(x)$ continuous at $x = 0$? Justify your answer using the definition of continuity.

$$h(x) = \begin{cases} g(x) & , \ x \neq 0, \frac{\pi}{2} \\ a & , \ x = 0 \end{cases}$$

Solution:

(a) $\displaystyle\lim_{x \to \frac{\pi}{2}^-} \frac{\sin x}{x(x - \pi/2)} = -\infty$ because as $x \to \frac{\pi}{2}^-$, we have $\sin x \to 1$ and $x(x - \pi/2) \to 0^-$. (That is, the denominator is negative as it approaches 0.)

$\displaystyle\lim_{x \to \frac{\pi}{2}^+} \frac{\sin x}{x(x - \pi/2)} = \infty$ because as $x \to \frac{\pi}{2}^+$, we have $\sin x \to 1$ and $x(x - \pi/2) \to 0^+$. (That is, the denominator is positive as it approaches 0.)

Since at least one of the preceding two limits is infinite (in fact, both are infinite for this function), $f(x)$ has a vertical asymptote at $\boxed{x = \pi/2}$.

(b)

$$\begin{aligned} \lim_{x \to 0} h(x) &= \lim_{x \to 0} \frac{\sin x}{x(x - \pi/2)} \\ &= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{(x - \pi/2)} \\ &= \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{1}{(x - \pi/2)} \end{aligned}$$

$$= (1)\left(\frac{1}{0 - \pi/2}\right)$$
$$= -\frac{2}{\pi}$$

The definition of continuity indicates that in order for $h(x)$ to be continuous at $x = 0$, $\lim_{x\to 0} h(x) = h(0)$. Since $h(0) = a$, in order for $h(x)$ to be continuous at $x = 0$ we must have $\boxed{a = -2/\pi}$.

5. (16 pts) A major movie studio has found that the function

$$P(t) = \frac{3t^3 - 9t}{\sqrt{4t^6 + 5t^4 + 5t}}$$

models their profit (in millions of dollars) from a certain movie $t > 0$ weeks after it was released.

(a) How many weeks after it is released does the movie studio "break even" on the movie? That is, when does $P(t) = 0$ for a realistic value of t?

(b) How much profit does the movie studio make in the long run? In other words, what is $\lim_{t\to\infty} P(t)$? Use correct units in your final answer.
(Reminder: You may not use L'Hopital's Rule or Dominance of Powers in any solutions on this exam.)

Solution:

(a) We know that $P(t) = 0$ when the numerator is equal to zero. Hence, we solve the equation

$$\begin{aligned} 3t^3 - 9t &= 0 \\ 3t(t^2 - 3) &= 0 \\ 3t(t - \sqrt{3})(t + \sqrt{3}) &= 0 \\ t &= 0, \pm\sqrt{3} \end{aligned}$$

We only want the positive value here, since this formula models profit t weeks *after* the movie was released. According to this formula, the movie studio's break-even point was $t = \sqrt{3}$ weeks after the movie was released.

(b) Computing the limit, we have

$$\begin{aligned} \lim_{t\to\infty} \frac{3t^3 - 9t}{\sqrt{4t^6 + 5t^4 + 5t}} &= \lim_{t\to\infty} \frac{t^3(3 - \frac{9}{t^2})}{t^3\sqrt{4 + \frac{5}{t^2} + \frac{5}{t^5}}} \\ &= \lim_{t\to\infty} \frac{3 - \frac{9}{t^2}}{\sqrt{4 + \frac{5}{t^2} + \frac{5}{t^5}}} \\ &= \frac{3 - 0}{\sqrt{4 + 0 + 0}} \\ &= \frac{3}{2}. \end{aligned}$$

This means that the movie studio expects to make $\$1.5$ million in the long-run.

6. (10 points) Correctly use a theorem to determine a closed interval in which $x + \tan x = 1$ has a solution. (Be sure to state the name of the theorem that is used and to clearly show that its hypotheses are satisfied.)

Solution:

This equation has a solution if and only if $x + \tan x - 1 = 0$ has a solution. Consider the function $f(x) = x + \tan x - 1$. It is sufficient to determine a closed interval where $f(x) = 0$. We note that $f(0) = -1$ and $f\left(\frac{\pi}{4}\right) = \frac{\pi}{4}$. Since f is continuous on $[0, \frac{\pi}{4}]$ and 0 is between $f(0)$ and $f\left(\frac{\pi}{4}\right)$, then the Intermediate Value Theorem guarantees the existence of some c in $[0, \frac{\pi}{4}]$ where $f(c) = 0$. So, $[0, \frac{\pi}{4}]$ is a closed interval where the given equation has a solution.

Note: There are infinitely many correct solutions to this problem. The above demonstrates one of them with correct justification. Note that $\tan x$ has infinite discontinuities at $x = n\pi + \frac{\pi}{2}$ for all integers n. So, an argument like the above will not work if the the end points of the interval have such an x-value between them.