1. (42 pts) The following problems are unrelated.

   (a) Find the derivative of $y = \sqrt{5x^2 - \sin x}$.

   (b) Evaluate $\displaystyle\int \frac{\arcsin(x)}{\sqrt{1-x^2}}\,dx$.

   (c) Evaluate $\displaystyle\int_0^{\ln\sqrt{3}} \frac{e^x}{1+e^{2x}}\,dx$.

   (d) Estimate the value of $\displaystyle\int_1^5 \ln\left(\frac{x}{x+1}\right)dx$ using a Riemann sum with right endpoints and $n = 4$ rectangles of equal width. Express your answer in terms of a single logarithm.

   (e) Evaluate $\displaystyle\lim_{x\to\infty} 2x\sinh\left(\frac{3}{x}\right)$.

Solution:

(a) $\dfrac{dy}{dx} = \dfrac{10x - \cos x}{2\sqrt{5x^2 - \sin x}}$.

(b) We will use the substitution $u = \arcsin(x)$. So, $du = \frac{1}{\sqrt{1-x^2}}dx$.

$$\begin{aligned}\int \frac{\arcsin(x)}{\sqrt{1-x^2}}\,dx &= \int u\,du \\ &= \frac{1}{2}u^2 + C \\ &= \frac{1}{2}\left(\arcsin(x)\right)^2 + C.\end{aligned}$$

(c) We will use the substitution $u = e^x$. So, $du = e^x\,dx$, the new upper limit of integration will be $u = e^{\ln\sqrt{3}} = \sqrt{3}$, and the new lower limit of integration will be $u = e^0 = 1$.

$$\begin{aligned}\int_0^{\ln\sqrt{3}} \frac{e^x}{1+e^{2x}}\,dx &= \int_1^{\sqrt{3}} \frac{1}{1+u^2}du \\ &= \arctan(\sqrt{3}) - \arctan(1) \\ &= \frac{\pi}{3} - \frac{\pi}{4} \\ &= \frac{\pi}{12}.\end{aligned}$$

(d)

$$\int_1^5 \ln\left(\frac{x}{x+1}\right)dx \approx \ln\frac{2}{3} + \ln\frac{3}{4} + \ln\frac{4}{5} + \ln\frac{5}{6} = \ln\frac{2}{6} = \ln\frac{1}{3} \text{ or } -\ln 3.$$

(e) If we rewrite the limit as $\displaystyle\lim_{x\to\infty} 2x\sinh\left(\frac{3}{x}\right) = \lim_{x\to\infty}\frac{2\sinh\left(\frac{3}{x}\right)}{\frac{1}{x}}$, then the limit is a $\frac{0}{0}$-indeterminate form and we can apply L'Hospital's Rule. So, we have

$$\begin{aligned}
\lim_{x\to\infty} 2x\sinh\left(\frac{3}{x}\right) &= \lim_{x\to\infty} \frac{2\sinh\left(\frac{3}{x}\right)}{\frac{1}{x}} \\
&=^H \lim_{x\to\infty} \frac{\cosh\left(\frac{3}{x}\right)\left(\frac{-6}{x^2}\right)}{\frac{-1}{x^2}} \\
&= \lim_{x\to\infty} 6\cosh\left(\frac{3}{x}\right) \\
&= 6.
\end{aligned}$$

2. (12 pts)

   (a) State the definition of continuity of a function, $f(x)$, at a point, $x = a$.

   (b) Now consider the function $f(x)$ defined on $[-1, 1]$ by

$$f(x) = \begin{cases} 2\sin^{-1}(x) & \text{if } x < \frac{1}{2} \\ c & \text{if } x = \frac{1}{2} \\ \cos^{-1}(x) & \text{if } x > \frac{1}{2}. \end{cases}$$

   Is there a value of c that makes f continuous at $x = \frac{1}{2}$? Justify your answer using the definition of continuity.

Solution:

   (a) A function, f, is **continuous at** $x = a$ if $\lim_{x\to a} f(x) = f(a)$.

   (b) We need $\lim_{x\to\frac{1}{2}} f(x) = f\left(\frac{1}{2}\right)$. In terms of one-sided limits, we need

$$\lim_{x\to\frac{1}{2}^-} f(x) = f\left(\frac{1}{2}\right) = \lim_{x\to\frac{1}{2}^+} f(x)$$

$$\lim_{x\to\frac{1}{2}^-} 2\sin^{-1}(x) = c = \lim_{x\to\frac{1}{2}^+} \cos^{-1}(x)$$

$$2\sin^{-1}\left(\frac{1}{2}\right) = c = \cos^{-1}\left(\frac{1}{2}\right)$$

$$2\cdot\frac{\pi}{6} = c = \frac{\pi}{3}$$

   So, the only value that will work is $c = \frac{\pi}{3}$.

3. (18 pts) Consider the function $h(x) = 2\sqrt{x} - \frac{1}{4}x$.

   (a) Locate all local extrema of $h(x)$. Clearly indicate which x-coordinates correspond to a local maximum (if any) and which correspond to a local minimum (if any).

   (b) Find $r'(x)$ when $r(x) = \int_2^{\tan x} h(t)\,dt$.

Solution:

(a) We see that $h'(x) = \frac{1}{\sqrt{x}} - \frac{1}{4}$. Since $h'(x)$ exists on the interior of the domain of $h(x)$ and $h'(x) = 0$ has only a solution of $x = 16$, then $x = 16$ is the only critical number of $h(x)$. Since $h'(x) > 0$ when $0 < x < 16$ and $h'(x) < 0$ when $x > 16$, then $h(x)$ has a local maximum value at $x = 16$, and no local minimum values.

(b) $r'(x) = \left(2\sqrt{\tan(x)} - \frac{1}{4}\tan(x)\right)\sec^2(x)$.

4. (32 pts) Consider $s(x) = \frac{e^{2x}}{3 - e^{2x}}$.

(a) Determine $s'(1)$. (Your final answer should be in terms of e.)

(b) Determine the inverse of $s(x)$. Be sure to label your final answer as $s^{-1}(x)$. (You may assume without proof that $s(x)$ is one-to-one.)

(c) Determine all horizontal asymptotes of $s(x)$. Justify each with the appropriate limit.

(d) Determine all vertical asymptotes of $s(x)$. Justify each with the appropriate limit.

Solution:

Note: For many of these problems, you may alternatively note that

$$s(x) = \frac{e^{2x}}{3 - e^{2x}} \cdot \frac{e^{-2x}}{e^{-2x}} = \frac{1}{3e^{-2x} - 1}$$

before proceeding. This will lead to solutions equivalent to the below.

(a)

$$\begin{aligned} s'(x) &= \frac{(3 - e^{2x})2e^{2x} - e^{2x}(-2e^{2x})}{(3 - e^{2x})^2} \\ &= \frac{6e^{2x}}{(3 - e^{2x})^2} \\ s'(1) &= \frac{6e^2}{(3 - e^2)^2}. \end{aligned}$$

(b)

$$\begin{aligned} y &= \frac{e^{2x}}{3 - e^{2x}} \\ x &= \frac{e^{2y}}{3 - e^{2y}} \\ (3 - e^{2y})x &= e^{2y} \\ e^{2y}(-x - 1) &= -3x \\ e^{2y} &= \frac{3x}{x + 1} \\ y &= \frac{1}{2}\ln\left(\frac{3x}{x + 1}\right) \\ s^{-1}(x) &= \frac{1}{2}\ln\left(\frac{3x}{x + 1}\right). \end{aligned}$$

(c) The following limits is an $\frac{\infty}{\infty}$-indeterminate form, so L'Hospital's rule applies:

$$\lim_{x\to\infty}\frac{e^{2x}}{3-e^{2x}} =^L \lim_{x\to\infty}\frac{2e^{2x}}{-2e^{2x}} = -1.$$

Note that L'Hospital's rule does not apply to the following limit:

$$\lim_{x\to-\infty}\frac{e^{2x}}{3-e^{2x}} = \frac{0}{3-0} = 0.$$

So, the horizontal asymptotes are $y=-1,0$.

(d) We see that

$$\lim_{x\to\ln\sqrt{3}^+}\frac{e^{2x}}{3-e^{2x}} = -\infty$$

because as $x\to\ln\sqrt{3}^+$, we see that e^{2x} approaches 3 and $3-e^{2x}$ approaches 0 from the negative numbers. Thus, there is a vertical asymptote at $x=\ln\sqrt{3}$. (Could have alternatively shown $\lim_{x\to\ln\sqrt{3}^-}\frac{e^{2x}}{3-e^{2x}}=\infty$ as a justification.)

5. (20 pts) Assume f and g are differentiable for all x, and we know the following values of f, f', g, and g'.

x	0	1	2	3	4
$f(x)$	-2	3	2	-4	0
$f'(x)$	1	3	5	-3	9
$g(x)$	-3	1	8	-1	1
$g'(x)$	0	3	1	-7	-2

(a) Determine the linearization of $y=f(x)$ at $x=3$.

(b) Use your answer from (a) to approximate $f(3.2)$.

(c) Determine $(f\circ g)'(4)$. (That is, find the derivative of $f(g(x))$ when evaluated at $x=4$.)

Solution:

(a)

$$\begin{aligned} L(x) &= f(3)+f'(3)(x-3) \\ &= -4-3(x-3). \end{aligned}$$

(b)

$$f(3.2)\approx L(3.2) = -4-3(3.2-3) = -4.6.$$

(c)

$$\begin{aligned} (f\circ g)'(4) &= f'(g(4))g'(4) \\ &= f'(1)(-2) \\ &= (3)(-2) \\ &= -6. \end{aligned}$$

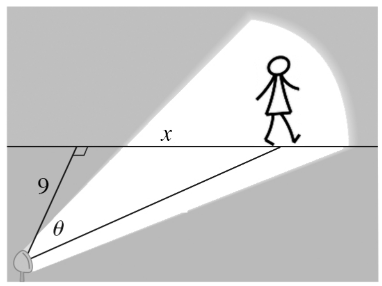
6. (26 points) The following problems are unrelated.

(a) The rate of change of atmospheric pressure, P, with respect to the elevation, h, is proportional to P. That is,

$$\frac{dP}{dh} = kP.$$

Assume the pressure at sea level is 18 pounds per square inch and the pressure at an elevation of 15,000 feet is 6 pounds per square inch. What would be the pressure at an elevation of 5,000 feet above sea level? (Remember to fully simplify your final answer. Specifically, there should be no e or natural logarithm in your final answer.)

(b) A girl walks along a straight path at a speed of 3 feet per second. A searchlight is located on the ground 9 feet from the path and is kept focused on the girl. At what rate is the searchlight rotating when the girl is 12 feet from the point on the path closest to the searchlight?



Solution:

(a) We know that $P(h) = P_0 e^{kh}$ where $P_0 = 18$ and $P(15,000) = 6$. We want to find $P(5,000)$. We must first use the given information to find k.

$$\begin{aligned} P(15,000) &= 6 \\ 18e^{15,000k} &= 6 \\ e^{15,000k} &= \frac{1}{3} \\ k &= \frac{\ln\left(\frac{1}{3}\right)}{15,000} \\ k &= -\frac{\ln 3}{15,000}. \end{aligned}$$

So,

$$\begin{aligned} P(5,000) &= 18e^{\left(-\frac{\ln 3}{15,000}\right)\cdot 5,000} \\ &= 18e^{\ln 3^{-1/3}} \\ &= 18(3)^{-1/3} \text{ pounds per square inch.} \end{aligned}$$

(b) We want to know $\frac{d\theta}{dt}$ when $\frac{dx}{dt} = 3$ feet per second and $x = 12$ feet. We can relate the variables x and θ by noting that

$$\tan\theta = \frac{x}{9}.$$

Differentiating, we see that

$$\sec^2(\theta)\frac{d\theta}{dt} = \frac{1}{9}\frac{dx}{dt}.$$

Solving for the desired rate and plugging in the known values at that moment, we have

$$\begin{aligned}\frac{d\theta}{dt} &= \frac{\cos^2(\theta)}{9}\frac{dx}{dt} \\ &= \frac{(9/15)^2}{9} \cdot 3 \\ &= \frac{3}{25} \text{ radians per second.}\end{aligned}$$

Note: We used the Pythagorean theorem to find the length of the hypotenuse and determine $\cos\theta$ at that moment.