1. (32 pts) The following problems are unrelated. For each, be sure to fully simplify your answers.

   (a) Evaluate $\displaystyle\int \frac{t^2+2}{t^{1/3}}\,dt$.

   (b) Evaluate $\displaystyle\int_{\pi^2/4}^{\pi^2} \frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}}\,dx$.

   (c) Suppose
   - $\displaystyle\int_0^2 r(x)\,dx = 6$,
   - $\displaystyle\int_{-1}^0 r(x)\,dx = 5$, and
   - $r(x)$ is an odd function.

   Use this information to evaluate $\displaystyle\int_{-2}^{-1} r(x)\,dx$.

   (d) Evaluate the following sum: $\displaystyle\sum_{i=1}^{n}\left[\left(2+\frac{i}{3n}\right)\frac{3}{n}\right]$. (Your final answer will be in terms of n but will not have any sigmas present.)

Solution:

(a)

$$\begin{aligned}\int \frac{t^2+2}{t^{1/3}}\,dt &= \int t^{5/3} + 2t^{-1/3}\,dt \\ &= \frac{3}{8}t^{8/3} + 3t^{2/3} + C.\end{aligned}$$

(b) We will use the substitution given by $u=\sqrt{x}$. It follows from this that $2\,du = \frac{1}{\sqrt{x}}dx$, and the new limits of integration will be $u=\pi/2$ and $u=\pi$:

$$\begin{aligned}\int_{\pi^2/4}^{\pi^2} \frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}}\,dx &= 2\int_{\pi/2}^{\pi}\cos(u)\,du \\ &= 2\left(\sin(\pi) - \sin\left(\frac{\pi}{2}\right)\right) \\ &= -2\end{aligned}$$

(c) Since $r(x)$ is odd, then we know that

$$\int_{-2}^{0} r(x)\,dx = -\int_0^2 r(x)\,dx = -6.$$

Thus, we have

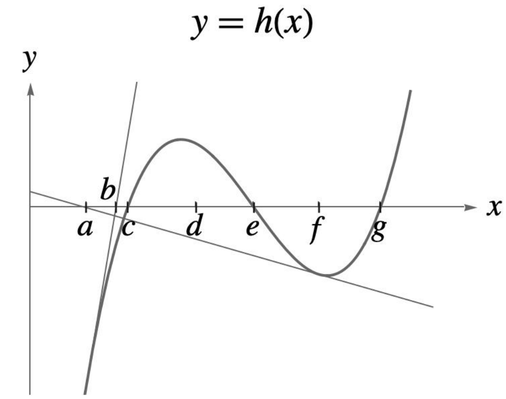
$$\int_{-2}^{-1} r(x)\,dx + \int_{-1}^{0} r(x)\,dx = \int_{-2}^{0} r(x)\,dx$$

$$\int_{-2}^{-1} r(x)\,dx + 5 = -6$$
$$\int_{-2}^{-1} r(x)\,dx = -11$$

(d)

$$\begin{aligned}\sum_{i=1}^{n}\left[\left(2+\frac{i}{3n}\right)\frac{3}{n}\right] &= \sum_{i=1}^{n}\left[\frac{6}{n}+\frac{i}{n^2}\right]\\ &= 6+\frac{1}{n^2}\sum_{i=1}^{n} i\\ &= 6+\frac{1}{n^2}\cdot\frac{n(n+1)}{2}\\ &= \frac{13}{2}+\frac{1}{2n}.\end{aligned}$$

2. (8 pts) Suppose Newton's Method is applied to the function $h(x)$, shown below, starting with an **initial approximation of x_1.** Tangent lines to the curve $y = h(x)$ corresponding to the first two iterations of Newton's Method are shown. No justification is necessary for the following two problems.



(a) Match the first three approximations x_1, x_2, and x_3 to the x-coordinates $a, b, c, d, e,\ f$, or g. Enter your answers in the table below.

(b) Assume that if Newton's Method continues, it will converge. To what value will it converge? Circle your answer in the space provided here. Circle only one answer.

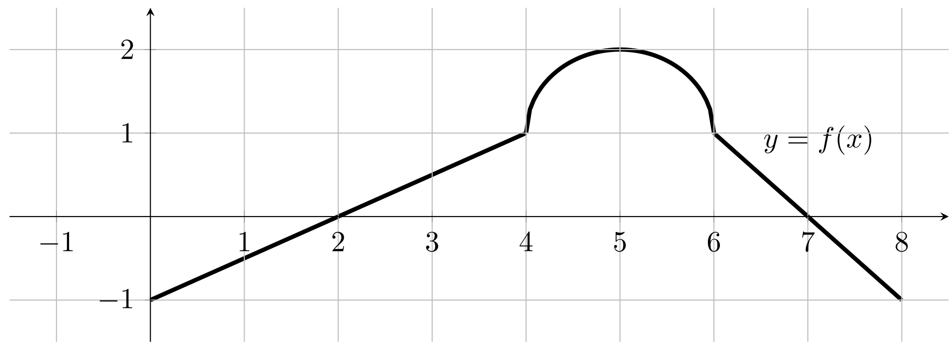
Solution:

(a) We have the following:

Approximation	a, b, c, d, e, f, or g
x_1 (initial guess)	f
x_2	a
x_3	b

(b) Newton's method will converge to c.

3. (24 pts) Consider the function $f(x)$ defined over $[0, 8]$ that is graphed below. It consists of two line segments and a semicircle.



(a) Find the average value of f over $[0, 8]$. (Express your answer in terms of π.)

(b) Approximate $\int_2^6 f(x)\,dx$ using four rectangles of equal width with the right endpoint rule.

(c) Let $g(x) = \int_0^x f(t)\,dt$. Find the x-coordinates of all inflection points of $g(x)$. (Provide a brief explanation.)

Solution:

(a) Using geometry (areas of triangles, a rectangle, and a semicircle) and remembering that area below the x-axis is negative, we see that $\int_0^8 f(x)\,dx = 2 + \frac{\pi}{2}$. So, the average value of f over $[0, 8]$ is

$$\frac{1}{8}\int_0^8 f(x)\,dx = \frac{1}{4} + \frac{\pi}{16}.$$

(b) Note that $\Delta x = 1$. So, the approximation is

$$\begin{aligned}(f(3) + f(4) + f(5) + f(6))\,\Delta x &= \left(\frac{1}{2} + 1 + 2 + 1\right) \cdot 1 \\ &= \frac{9}{2}.\end{aligned}$$

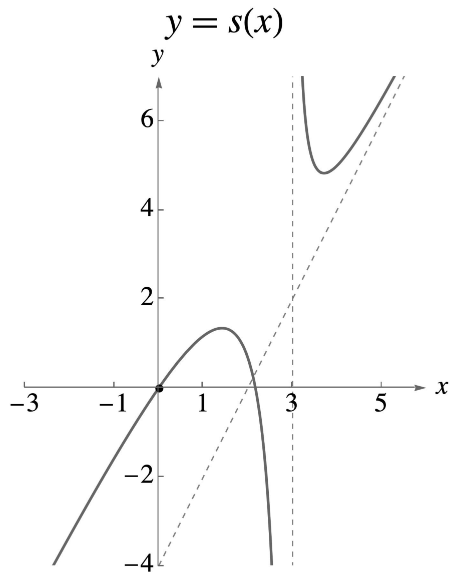
(c) From the Fundamental Theorem of Calculus (Part 1), we know that $g'(x) = f(x)$. Therefore, $g''(x) = f'(x)$. From the graph, we know that $g''(x)$ will only have a sign-change when $x = 5$. Also, since g is differentiable on $(0, 8)$, we know that it is continuous on $(0, 8)$. Thus, the only inflection point for $y = g(x)$ occurs at $x = 5$.

4. (15 pts) Using the grid below, sketch the graph of a single function, $y = s(x)$ with each of the following characteristics. (Sketch dashed lines to indicate any asymptotes that are present. The concavity of your graph should be clear.)

- s is continuous for all $x \neq 3$
- $\lim_{x \to 3^+} s(x) = \infty$
- $\lim_{x \to 3^-} s(x) = -\infty$
- $s''(x) < 0$ only for x in $(-\infty, 3)$
- $s(0) = 0$

- $\lim_{x\to\infty} (s(x) - (2x-4)) = 0$

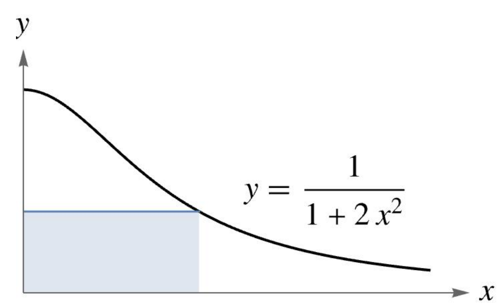
Solution: Here is the graph of a function that satisfies all of the stated conditions:



5. (21 pts) The following problems are unrelated.

   (a) A particle has acceleration given by $a(t) = 4t - 10$ meters per second squared at t seconds. It has a velocity of 4 meters per second at $t = 1$ seconds. Given this, at what time(s) is the particle at rest? Be sure to fully justify your answers.

   (b) The rectangle shown has one side on the positive x-axis, one side on the positive y-axis, and its upper right corner on the curve $y = \dfrac{1}{1+2x^2}$. What dimensions give the rectangle its largest area? (Be sure to justify you have found the absolute maximum.)



Solution:

   (a) We want to solve $v(t) = 0$. First, we must find $v(t)$. We know that $v'(t) = 4t - 10$. So, $v(t) = 2t^2 - 10t + C$. To find C, we note that $v(1) = 4$. So,

$$4 = v(1) = 2(1)^2 - 10(1) + C$$

   implies $C = 12$.

   Now, we solve the equation:

$$\begin{aligned} v(t) &= 0 \\ 2t^2 - 10t + 12 &= 0 \\ 2(t-2)(t-3) &= 0 \end{aligned}$$

Thus, the particle is at rest at $t = 2, 3$ seconds.

(b) The area of the rectangle is given by $A = xy$ where (x, y) are the coordinates of the upper right corner. Since this point lies on the curve, then we see that $A(x) = \frac{x}{1+2x^2}$. We want to find the absolute maximum of this function over $(0, \infty)$.

If it exists, the absolute maximum must occur at a critical number. We see that $A'(x) = \frac{1-2x^2}{(1+2x^2)^2}$ exists on $(0, \infty)$, but equals 0 when $x = \pm\frac{1}{\sqrt{2}}$. Of these, only $x = \frac{1}{\sqrt{2}}$ lies in our interval.

Since $A'(x) > 0$ where $0 < x < \frac{1}{\sqrt{2}}$ and $A'(x) < 0$ where $x > \frac{1}{\sqrt{2}}$, then we know there is an absolute maximum value of $A(x)$ at $x = \frac{1}{\sqrt{2}}$. For this x-value, we have $y = \frac{1}{2}$. Thus, the dimensions that give the largest area are a width of $\frac{1}{\sqrt{2}}$ and a height of $\frac{1}{2}$.