1. (30 pts) The following problems are unrelated.
   (a) Find the tangent line of $y = \dfrac{x}{x+1}$ at $x = 1$.
   (b) Find the derivative of $p(x) = \sin^2(x) + \sec\left(x^3\right)$. (Please do not simplify your final answer.)
   (c) Find $\frac{dy}{dx}$ at $(0, 1)$ when $(x - y)^5 = (x - 1)y^3$.

Solution:

(a) The point of tangency is $\left(1, \frac{1}{2}\right)$. To find the slope, we need the derivative:

$$y' = \frac{(x+1) - x}{(x+1)^2} = \frac{1}{(x+1)^2}.$$

So, we see the slope is $y'(1) = \frac{1}{4}$. Thus, the tangent line is

$$y = \frac{1}{2} + \frac{1}{4}(x - 1).$$

(b)

$$p'(x) = 2\sin(x)\cos(x) + 3x^2 \sec(x^3)\tan(x^3).$$

(c) First, we differentiate both sides:

$$5(x-y)^4\left(1 - \frac{dy}{dx}\right) = 3(x-1)y^2\frac{dy}{dx} + y^3.$$

Then, we need to algebraically solve for $\frac{dy}{dx}$:

$$\begin{aligned}
5(x-y)^4 - 5(x-y)^4\frac{dy}{dx} &= 3(x-1)y^2\frac{dy}{dx} + y^3 \\
5(x-y)^4 - y^3 &= 5(x-y)^4\frac{dy}{dx} + 3(x-1)y^2\frac{dy}{dx} \\
5(x-y)^4 - y^3 &= \left(5(x-y)^4 + 3(x-1)y^2\right)\frac{dy}{dx} \\
\frac{dy}{dx} &= \frac{5(x-y)^4 - y^3}{5(x-y)^4 + 3(x-1)y^2}
\end{aligned}$$

Lastly, we evaluate the derivative at $(0, 1)$:

$$\frac{dy}{dx}\Big|_{(0,1)} = \frac{5(0-1)^4 - 1^3}{5(0-1)^4 + 3(0-1)(1)^2} = 2.$$

2. (22 pts) The following problems are unrelated.
   (a) Consider $r(x) = \tan(x)$.
      (i) Determine the linearization of $r(x)$ at $x = \frac{\pi}{4}$.
      (ii) Use your answer from (i) to approximate $\tan(1)$. (Your final answer should be in terms of π.)

(iii) Is your approximation from (ii) an overestimate or an underestimate? Explain your answer.

(b) Suppose f is a function where the following are all true:

- $f'(x) = \dfrac{x^2}{1+2f(x)}$,
- $f(1) = 2$, and
- $f'(x)$ is differentiable at $x = 1$.

Using this information, determine $f''(1)$.

Solution:

(a) (i) We see that $r'(x) = \sec^2(x)$, so $r'\left(\frac{\pi}{4}\right) = 2$. Also, we have $r\left(\frac{\pi}{4}\right) = 1$. Thus, the linearization is

$$L(x) = 1 + 2\left(x - \frac{\pi}{4}\right).$$

(ii)

$$\begin{aligned} r(1) &\approx L(1) \\ &= 1 + 2\left(1 - \frac{\pi}{4}\right) \\ &= 3 - \frac{\pi}{2}. \end{aligned}$$

(iii) $r''(x) = 2\sec^2(x)\tan(x)$ is positive for $x > 0$, so $r(x)$ is concave up for $x > 0$. Thus, the curve is above the linearization and the approximation must be an underestimate.

(b) First, we differentiate (using the quotient rule):

$$f''(x) = \frac{(1+2f(x))(2x) - 2x^2 f'(x)}{(1+2f(x))^2}.$$

Next we need to evaluate this at $x = 1$. Note that we also need $f'(1)$ to do this. So, we have

$$f'(1) = \frac{1^2}{1+2f(1)} = \frac{1}{5}$$

which tells us that

$$\begin{aligned} f''(1) &= \frac{(1+2f(1))(2(1)) - 2(1)^2 f'(1)}{(1+2f(1))^2} \\ &= \frac{(1+2(2))(2) - 2\left(\frac{1}{5}\right)}{(1+2(2))^2} \\ &= \frac{48}{125}. \end{aligned}$$

3. (28 pts) The following problems are unrelated.

(a) Use the definition of the derivative to show that the derivative of $g(x) = 5x^2 - 4x$ is $g'(x) = 10x - 4$.

(b) Consider $s(x) = (x-3)^{-1}$. Show there is **no** value c in $(0, 6)$ such that $s'(c) = \dfrac{s(6) - s(0)}{6}$. Why does this not contradict the Mean Value Theorem?

(c) Consider $r(x) = x(x+8)^{\frac{1}{3}}$. Determine all critical numbers of $r(x)$.

Solution:

(a)

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0} \frac{(5(x+h)^2-4(x+h))-(5x^2-4x)}{h} \\ &= \lim_{h\to 0} \frac{(5x^2+10xh+5h^2-4x-4h))-(5x^2-4x)}{h} \\ &= \lim_{h\to 0} \frac{10xh+5h^2-4h}{h} \\ &= \lim_{h\to 0} \frac{h(10x+5h-4)}{h} \\ &= \lim_{h\to 0} 10x+5h-4 \\ &= 10x-4. \end{aligned}$$

(b) Note that $\frac{s(6)-s(0)}{6} = \frac{1}{9} > 0$ but $s'(x) = -\frac{1}{(x-3)^2} < 0$ for all x. So, no value c in $[0,6]$ such that $s'(c) = \dfrac{s(6)-s(0)}{6}$. This does not contradict the Mean Value Theorem because $s(x)$ is not continuous on $[0,6]$ (because it is not continuous at $x=3$) and $s(x)$ is not differentiable at $(0,6)$ (because it is not differentiable at $x=3$).

(c) We need to determine the values in the domain of $r(x)$ where $r'(x)=0$ and where $r'(x)$ does not exist. Note that $r(x)$ is defined for all x.

$$\begin{aligned} r'(x) &= x\cdot\frac{1}{3}(x+8)^{\frac{-2}{3}} + (x+8)^{\frac{1}{3}} \\ &= \frac{4(x+6)}{3(x+8)^{\frac{2}{3}}}. \end{aligned}$$

We see that $r'(x)=0$ has a solution of $x=-6$ and $r'(x)$ does not exist at $x=-8$. So, the critical numbers of $r(x)$ are $x=-8,-6$.

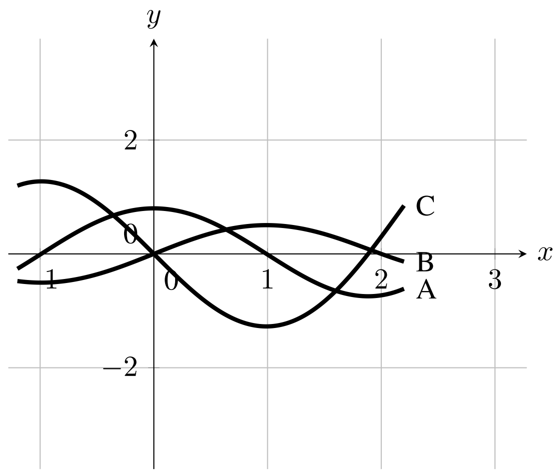
4. (20 pts) The following problems are unrelated.

(a) A gloopy is an intelligent species that lives on the planet Blorpy. A gloopy's volume, V, is always the product of their wingspan squared, W^2, and their foot length, F:

$$V = W^2F.$$

Additionally, a gloopy's foot length grows at a constant rate of 2 cm/year, and their wingspan grows at half that rate. How quickly is the volume of a gloopy increasing if its current foot length is 6 cm and its wingspan is 20 cm?

(b) Graphed below are $y=f(x)$, $y=f'(x)$, and $y=f''(x)$. Match each of these to the correct label of A, B, or C. Place your answer in the table below. (No justification is required.)



Function	A, B, or C
$y = f(x)$	
$y = f'(x)$	
$y = f''(x)$	

Solution:

(a) Our goal is to find $\frac{dV}{dt}$ at the moment when a gloopy's current foot length is 6 cm and its wingspan is 20 cm. First, we differentiate the given formula with respect to time, t:

$$\begin{aligned} \frac{d}{dt}(V) &= \frac{d}{dt}\left(W^2 F\right) \\ \frac{dV}{dt} &= W^2 \frac{dF}{dt} + 2WF\frac{dW}{dt}. \end{aligned}$$

Now, we can plug in these values to obtain the desired rate at that moment:

$$\begin{aligned} \frac{dV}{dt} &= (20 \text{ cm})^2 (2 \text{ cm/year}) + 2(20 \text{ cm })(6\text{cm})(1 \text{ cm/year}) \\ &= 1040 \text{ cubic centimeters per year }. \end{aligned}$$

(b) At $x = 0$, we see that the function for A must have as its derivative the function for C. Likewise, at $x = 1$, we see that the function for A is the derivative for the function for B. So, we have

Function	A, B, or C
$y = f(x)$	B
$y = f'(x)$	A
$y = f''(x)$	C