1. (30 pts)
Evaluate the following limits and simplify your answers. If a limit does not exist, clearly state this.
(Reminder: You may not use L’Hopital’s Rule or Dominance of Powers in any solutions on this exam.)

(a) \( \lim_{x \to -5} \frac{|x^2 - 25|}{2x + 10} \)
(b) \( \lim_{x \to 0} \frac{5x - \tan(x)}{x} \)
(c) \( \lim_{x \to 0} \left( x^6 \sin \left( \frac{\pi}{x} \right) - 3 \right) \)

Solution:

(a) First, we note that \( \frac{|x^2 - 25|}{2x + 10} = \frac{|x - 5||x + 5|}{2(x + 5)} \), and that the behavior of the function on either side of \( x = -5 \) may vary. So, we consider the corresponding one-sided limits:

\[
\lim_{x \to -5^+} \frac{|x^2 - 25|}{2x + 10} = \lim_{x \to -5^+} \frac{|x - 5||x + 5|}{2(x + 5)} = \lim_{x \to -5^+} \frac{-(x - 5)(x + 5)}{2(x + 5)} = \lim_{x \to -5^+} \frac{-(x - 5)}{2} = 5
\]

and

\[
\lim_{x \to -5^-} \frac{|x^2 - 25|}{2x + 10} = \lim_{x \to -5^-} \frac{|x - 5||x + 5|}{2(x + 5)} = \lim_{x \to -5^-} \frac{(x - 5)(x + 5)}{2(x + 5)} = \lim_{x \to -5^-} \frac{x - 5}{2} = -5
\]

Since the two one-sided limit disagree, then we know that \( \lim_{x \to -5} \frac{|x^2 - 25|}{2x + 10} \) does not exist.

(b) Recall that \( \lim_{x \to 0} \frac{\sin(x)}{x} = 1 \) and that trigonometric functions, like \( \cos(x) \), are continuous on their domains. So,

\[
\lim_{x \to 0} \frac{5x - \tan(x)}{x} = \lim_{x \to 0} \frac{5x}{x} - \lim_{x \to 0} \frac{\tan(x)}{x} = \lim_{x \to 0} 5 - \left( \lim_{x \to 0} \frac{\sin(x)}{x} \right) \left( \lim_{x \to 0} \frac{1}{\cos(x)} \right) = 5 - 1(1) = 4.
\]
(c) \( \lim_{x \to 0} \left( x^6 \sin \left( \frac{\pi}{x} \right) - 3 \right) \)

To evaluate this limit, we will make use of the Squeeze Theorem. Note that \(-1 \leq \sin \left( \frac{\pi}{x} \right) \leq 1\) for \(x \neq 0\). Multiplying all sides by \(x^6\), we have

\[-x^6 \leq x^6 \sin \left( \frac{\pi}{x} \right) \leq x^6.\]

If we then subtract 3 from all sides, we have

\[-x^6 - 3 \leq x^6 \sin \left( \frac{\pi}{x} \right) - 3 \leq x^6 - 3.\]

Noting that

\[
\lim_{x \to 0} (-x^6 - 3) = -3 = \lim_{x \to 0} (x^6 - 3)
\]

we then have that the Squeeze Theorem implies

\[
\lim_{x \to 0} \left( x^6 \sin \left( \frac{\pi}{x} \right) - 3 \right) = -3.
\]

2. (20 pts) Problems (a) and (b) are not related.

(a) Suppose \(\csc(\theta) = 5\) and \(\cos(\theta) < 0\). Determine \(\tan(2\theta)\).

(b) Consider \(f(x) = \cos(2x)\) and \(g(x) = 3\cos(x) - 2\). Find all values of \(x\) on the interval \([0, 2\pi]\) that satisfy \(f(x) = g(x)\).

Solution:

(a) Note that

\[
\tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)} = \frac{2\sin(\theta)\cos(\theta)}{1 - 2\sin^2(\theta)}.
\]

We know that \(\sin(\theta) = \frac{1}{\csc(\theta)} = \frac{1}{5}\). So, we have \(\cos(\theta) = -\sqrt{1 - \sin^2(\theta)} = -\frac{\sqrt{24}}{5}\).

Combining our observations, we have

\[
\tan(2\theta) = \frac{2 \left( \frac{1}{5} \right) \left( -\frac{\sqrt{24}}{5} \right)}{1 - 2 \left( \frac{1}{25} \right)} = -\frac{2\sqrt{24}}{23}.
\]

(b) We apply a double angle identity and algebraically manipulate:

\[
f(x) = g(x) \\
\cos(2x) = 3\cos(x) - 2 \\
2\cos^2(x) - 1 = 3\cos(x) - 2 \\
2\cos^2(x) - 3\cos(x) + 1 = 0 \\
(2\cos(x) - 1)(\cos(x) - 1) = 0
\]

Using the zero product property, we see that we need the solutions in \([0, 2\pi]\) for \(\cos(x) = \frac{1}{2}\) and \(\cos(x) = 1\). Using the unit circle, we have solutions

\[x = 0, \frac{\pi}{3}, \frac{5\pi}{3}, 2\pi.\]
3. (20 pts)

Consider

\[ f(x) = \begin{cases} 
\frac{2}{x}, & x < a \\
2x + 3, & x \geq a 
\end{cases} \]

(a) Determine all values for \( a \) such that \( f(x) \) will be continuous for all \( x \). (Be sure to justify your answer with the definition of continuity.)

(b) Use the grid provided on the next page to sketch a graph for \( y = f(x) \) using one of the values of \( a \) you found in (a). (Clearly state the value of \( a \) being used. Be sure your axes and any intercepts are clearly labeled.)

Solution:

(a) Note that we need \( a < 0 \), otherwise \( f \) will have an infinite discontinuity at \( x = 0 \). Assuming \( a < 0 \), we see that \( f \) is defined for all \( x \) and will always be continuous when \( x \neq a \) because \( f \) will be a rational function. At \( x = a \), we need

\[
\lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a^+} f(x)
\]

\[
\lim_{x \to a^-} \frac{2}{x} = 2a + 3 = \lim_{x \to a^+} 2x + 3
\]

\[
\frac{2}{a} = 2a + 3 = 2a + 3
\]

\[
0 = 2a^2 + 3a - 2
\]

\[
0 = (2a - 1)(a + 2).
\]

This is only the case if \( a = \frac{1}{2} \) or \( a = -2 \). Since we already noted that \( a < 0 \), then we are left with a single value of \( a = -2 \).

(b) For \( a = -2 \), we have the following graph of \( y = f(x) \):

4. (30 pts) The following problems are not related.
(a) Consider \( s(x) = 3x^3 - 2x^2 \) and \( r(x) = 2x + 5 \). Use a theorem to determine an interval where \( s(x) = r(x) \) for some \( x \) on that interval. (Clearly state the name of the theorem used and be sure to justify its use.)

(b) Consider \( h(x) = \frac{4 + \sin(x)}{7 - 3x} \). Determine all vertical asymptotes of \( y = h(x) \). Justify each with the definition of vertical asymptotes.

(c) Use the grid provided on the next page to sketch the graph of a single function \( y = f(x) \) that satisfies each of the following conditions. (Be sure that your axes and all relevant points are clearly labeled.)

\[
\begin{align*}
\lim_{x \to -\infty} f(x) &= -3 \\
\lim_{x \to 2^+} f(x) &= 2 \\
\lim_{x \to 2^-} f(x) &= \infty \\
f(2) &= -1 \\
f(-1) &= 0 \\
f \text{ is an odd function.}
\end{align*}
\]

Solution:

(a) Let \( g(x) = s(x) - r(x) = 3x^3 - 2x^2 - 2x - 5 \). Note that \( s(x) = r(x) \) if and only if \( g(x) = 0 \). So, we will apply the Intermediate Value Theorem to show that \( g(x) = 0 \) on some interval. Note that \( g(x) \) is continuous everywhere because it is a polynomial. Evaluating the function at some different values, we see that

\[
\begin{align*}
g(1) &= -6 < 0 \\
g(2) &= 7 > 0.
\end{align*}
\]

Since \( g \) is continuous on \([1, 2]\) and 0 is between \( g(1) \) and \( g(2) \), then \( g(c) = 0 \) for some \( c \) in \([1, 2]\). Thus, \( s(c) = r(c) \) for some \( c \) in \([1, 2]\). (Note that we could have evaluated \( g \) at different values and potentially found a different interval that would also be correct.)

(b) The only \( x \)-value where a vertical asymptote may occur is when \( 7 - 3x = 0 \). That is, at \( x = \frac{7}{3} \). So, let’s consider one of the one-sided limits as \( x \to \frac{7}{3} \). (We only need to consider one of them.)

For the right side, we have

\[
\lim_{x \to \frac{7}{3}^+} \frac{4 + \sin(x)}{7 - 3x} = -\infty
\]

because as \( x \) approaches \( \frac{7}{3} \) from the right, the numerator is between 3 and 5 (therefore positive), but the denominator is approaching 0 and is negative as it does so.

For the left side, we have

\[
\lim_{x \to \frac{7}{3}^-} \frac{4 + \sin(x)}{7 - 3x} = \infty
\]

because as \( x \) approaches \( \frac{7}{3} \) from the left, the numerator is between 3 and 5 (therefore positive), but the denominator is approaching 0 and is positive as it does so.

Either one of these limits (with justification) is sufficient to prove that \( y = h(x) \) has a vertical asymptote at \( x = \frac{7}{3} \).

(c) Below is a graph of one potential answer to this problem. There are many correct answers.
$y = f(x)$