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1. (30 pts)

   Evaluate the following limits and simplify your answers. If a limit does not exist, clearly state this.
   *(Reminder: You may not use L'Hopital's Rule or Dominance of Powers in any solutions on this exam.)*

   (a) $\lim_{x \to -5} \frac{|x^2 - 25|}{2x + 10}$

   (b) $\lim_{x \to 0} \frac{5x - \tan(x)}{x}$

   (c) $\lim_{x \to 0} \left( x^6 \sin\left(\frac{\pi}{x}\right) - 3 \right)$

   **Solution:**

   (a) First, we note that $\frac{|x^2-25|}{2x+10} = \frac{|x-5||x+5|}{2(x+5)}$, and that the behavior of the function on either side of $x = -5$ may vary. So, we consider the corresponding one-sided limits:

$$\begin{aligned}\lim_{x \to -5^+} \frac{|x^2 - 25|}{2x + 10} &= \lim_{x \to -5^+} \frac{|x-5||x+5|}{2(x+5)} \\ &= \lim_{x \to -5^+} \frac{-(x-5)(x+5)}{2(x+5)} \\ &= \lim_{x \to -5^+} \frac{-(x-5)}{2} \\ &= 5\end{aligned}$$

   and

$$\begin{aligned}\lim_{x \to -5^-} \frac{|x^2 - 25|}{2x + 10} &= \lim_{x \to -5^-} \frac{|x-5||x+5|}{2(x+5)} \\ &= \lim_{x \to -5^-} \frac{(x-5)(x+5)}{2(x+5)} \\ &= \lim_{x \to -5^-} \frac{x-5}{2} \\ &= -5\end{aligned}$$

   Since the two one-sided limit disagree, then we know that $\lim_{x \to -5} \frac{|x^2 - 25|}{2x + 10}$ does not exist.

   (b) Recall that $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$ and that trigonometric functions, like $\cos(x)$, are continuous on their domains. So,

$$\begin{aligned}\lim_{x \to 0} \frac{5x - \tan(x)}{x} &= \lim_{x \to 0} \frac{5x}{x} - \lim_{x \to 0} \frac{\tan(x)}{x} \\ &= \lim_{x \to 0} 5 - \left( \lim_{x \to 0} \frac{\sin(x)}{x} \right) \left( \lim_{x \to 0} \frac{1}{\cos(x)} \right) \\ &= 5 - 1(1) \\ &= 4.\end{aligned}$$

(c) $\lim_{x\to 0}\left(x^6 \sin\left(\frac{\pi}{x}\right) - 3\right)$

To evaluate this limit, we will make use of the Squeeze Theorem.
Note that $-1 \le \sin\left(\frac{\pi}{x}\right) \le 1$ for $x \ne 0$. Multiplying all sides by $x^6$, we have

$$-x^6 \le x^6 \sin\left(\frac{\pi}{x}\right) \le x^6.$$

If we then subtract 3 from all sides, we have

$$-x^6 - 3 \le x^6 \sin\left(\frac{\pi}{x}\right) - 3 \le x^6 - 3.$$

Noting that

$$\lim_{x\to 0}(-x^6 - 3) = -3 = \lim_{x\to 0}(x^6 - 3)$$

, we then have that the Squeeze Theorem implies $\lim_{x\to 0}\left(x^6 \sin\left(\frac{\pi}{x}\right) - 3\right) = -3.$

2. (20 pts) Problems (a) and (b) are not related.

   (a) Suppose $\csc(\theta) = 5$ and $\cos(\theta) < 0$. Determine $\tan(2\theta)$.

   (b) Consider $f(x) = \cos(2x)$ and $g(x) = 3\cos(x) - 2$. Find all values of $x$ on the interval $[0, 2\pi]$ that satisfy $f(x) = g(x)$.

**Solution:**

(a) Note that

$$\begin{aligned}\tan(2\theta) &= \frac{\sin(2\theta)}{\cos(2\theta)}\\ &= \frac{2\sin(\theta)\cos(\theta)}{1 - 2\sin^2(\theta)}.\end{aligned}$$

We know that $\sin(\theta) = \frac{1}{\csc(\theta)} = \frac{1}{5}$. So, we have $\cos(\theta) = -\sqrt{1 - \sin^2(\theta)} = -\frac{\sqrt{24}}{5}$.
Combining our observations, we have

$$\begin{aligned}\tan(2\theta) &= \frac{2\left(\frac{1}{5}\right)\left(-\frac{\sqrt{24}}{5}\right)}{1 - 2\left(\frac{1}{25}\right)}\\ &= -\frac{2\sqrt{24}}{23}.\end{aligned}$$

(b) We apply a double angle identity and algebraically manipulate:

$$\begin{aligned}f(x) &= g(x)\\ \cos(2x) &= 3\cos(x) - 2\\ 2\cos^2(x) - 1 &= 3\cos(x) - 2\\ 2\cos^2(x) - 3\cos(x) + 1 &= 0\\ (2\cos(x) - 1)(\cos(x) - 1) &= 0\end{aligned}$$

Using the zero product property, we see that we need the solutions in $[0, 2\pi]$ for $\cos(x) = \frac{1}{2}$ and $\cos(x) = 1$.
Using the unit circle, we have solutions

$$x = 0, \frac{\pi}{3}, \frac{5\pi}{3}, 2\pi.$$

3. (20 pts)

Consider

$$f(x) = \begin{cases} \frac{2}{x}, & x < a \\ 2x+3, & x \geq a \end{cases}$$

(a) Determine all values for $a$ such that $f(x)$ will be continuous for all $x$. (Be sure to justify your answer with the definition of continuity.)

(b) Use the grid provided on the next page to sketch a graph for $y = f(x)$ using one of the values of $a$ you found in (a). (Clearly state the value of $a$ being used. Be sure your axes and any intercepts are clearly labeled.)

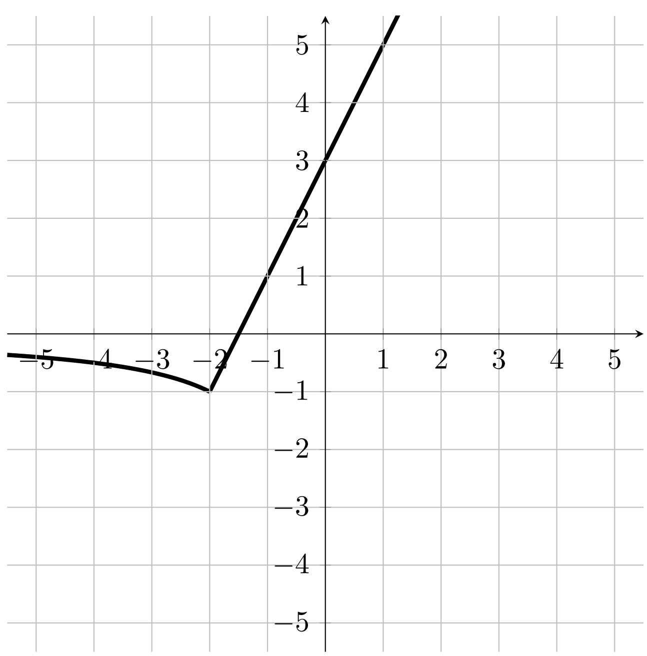
**Solution:**

(a) Note that we need $a < 0$, otherwise $f$ will have an infinite discontinuity at $x = 0$. Assuming $a < 0$, we see that $f$ is defined for all $x$ and will always be continuous when $x \neq a$ because $f$ will be a rational function. At $x = a$, we need

$$\begin{aligned} \lim_{x\to a^-} f(x) &= f(a) = \lim_{x\to a^+} f(x) \\ \lim_{x\to a^-} \frac{2}{x} &= 2a+3 = \lim_{x\to a^+} 2x+3 \\ \frac{2}{a} &= 2a+3 = 2a+3 \\ 0 &= 2a^2+3a-2 \\ 0 &= (2a-1)(a+2). \end{aligned}$$

This is only the case if $a = \frac{1}{2}$ or $a = -2$. Since we already noted that $a < 0$, then we are left with a single value of $a = -2$.

(b) For $a = -2$, we have the following graph of $y = f(x)$:




4. (30 pts) The following problems are not related.

(a) Consider $s(x) = 3x^3 - 2x^2$ and $r(x) = 2x + 5$. Use a theorem to determine an interval where $s(x) = r(x)$ for some $x$ on that interval. (Clearly state the name of the theorem used and be sure to justify its use.)

(b) Consider $h(x) = \dfrac{4 + \sin(x)}{7 - 3x}$. Determine all vertical asymptotes of $y = h(x)$. Justify each with the definition of vertical asymptotes.

(c) Use the grid provided on the next page to sketch the graph of a single function $y = f(x)$ that satisfies each of the following conditions. (Be sure that your axes and all relevant points are clearly labeled.)

$$\lim_{x \to -\infty} f(x) = -3 \qquad \lim_{x \to 2^+} f(x) = 2 \qquad \lim_{x \to 2^-} f(x) = \infty$$

$$f(2) = -1 \qquad f(-1) = 0 \qquad f \text{ is an odd function.}$$

**Solution:**

(a) Let $g(x) = s(x) - r(x) = 3x^3 - 2x^2 - 2x - 5$. Note that $s(x) = r(x)$ if and only if $g(x) = 0$. So, we will apply the Intermediate Value Theorem to show that $g(x) = 0$ on some interval. Note that $g(x)$ is continuous everywhere because it is a polynomial. Evaluating the function at some different values, we see that

$$g(1) = -6 < 0$$
$$g(2) = 7 > 0.$$

Since $g$ is continuous on $[1, 2]$ and 0 is between $g(1)$ and $g(2)$, then $g(c) = 0$ for some $c$ in $[1, 2]$. Thus, $s(c) = r(c)$ for some $c$ in $[1, 2]$. (Note that we could have evaluated $g$ are different values and potentially found a different interval that would also be correct.)

(b) The only $x$-value where a vertical asymptote may occur is when $7 - 3x = 0$. That is, at $x = \dfrac{7}{3}$. So, let's consider one of the one-sided limits as $x \to \frac{7}{3}$. (We only need to consider one of them.)
For the right side, we have

$$\lim_{x \to \frac{7}{3}^+} \frac{4 + \sin(x)}{7 - 3x} = -\infty$$

because as $x$ approaches $\frac{7}{3}$ from the right, the numerator is between 3 and 5 (therefore positive), but the denominator is approaching 0 and is negative as it does so.
For the left side, we have

$$\lim_{x \to \frac{7}{3}^-} \frac{4 + \sin(x)}{7 - 3x} = \infty$$

because as $x$ approaches $\frac{7}{3}$ from the left, the numerator is between 3 and 5 (therefore positive), but the denominator is approaching 0 and is positive as it does so.
Either one of these limits (with justification) is sufficient to prove that $y = h(x)$ has a vertical asymptote at $x = \frac{7}{3}$.

(c) Below is a graph of **one** potential answer to this problem. There are many correct answers.

$$y = f(x)$$