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1. (24 pts) Consider

$$f(x) = \begin{cases} 2x+1 & 0 \le x \le 1 \\ 4 - x^2 & 1 < x \le 2 \end{cases}.$$

   (a) Sketch a graph of $y = f(x)$. (The axes should be clearly labeled.)

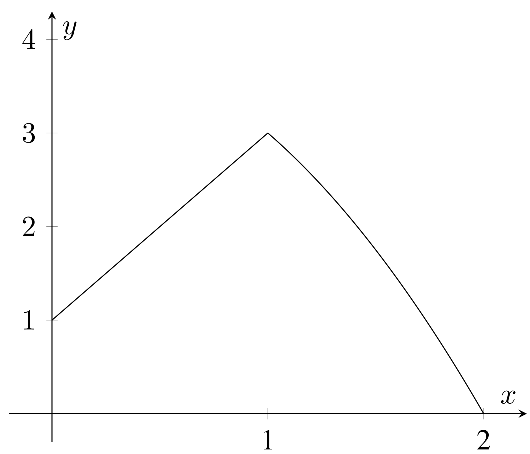
   (b) Is f differentiable at $x = 1$? (Justify your answer with the definition of the derivative.)

   (c) Does the Mean Value Theorem apply to $f(x)$ on $[0, 2]$? Why or why not? (Note: A complete answer will include a statement of the Mean Value Theorem and which hypotheses are satisfied or not.)

   (d) Evaluate $\int_0^2 f(x)\,dx$

Solution:

(a)



(b) No. Note that

$$\lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^-} \frac{2(1+h) + 1 - 3}{h} = 2$$

but

$$\lim_{h0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h0^+} \frac{4 - (1+h)^2 - 3}{h} = -2.$$

So, $f'(1) = \lim_{h0} \dfrac{f(1+h) - f(1)}{h}$ does not exist.

(c) Recall that the Mean Value Theorem states "If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$." For our example, f is continuous on $[0, 2]$ but it is not differentiable on $(0, 2)$. So, no, the mean value theorem does not apply.

(d)

$$\begin{aligned} \int_0^2 f(x)\,dx &= \int_0^1 2x + 1\,dx + \int_1^2 4 - x^2\,dx \\ &= \left[x^2 + x\right]_0^1 + \left[4x - \frac{1}{3}x^3\right]_1^2 \\ &= \frac{11}{3} \end{aligned}$$

2. (22 pts) The following problems are not related.

   (a) The planet N'Var has acceleration due to gravity of -12 meters per second squared. Suppose a stone is thrown from the top of a 42 meter tall building on the planet N'Var with an initial upward velocity of 36 meters per second. What is the velocity of the stone when it strikes the ground?

   (b) Determine the absolute minimum and the absolute maximum of $f(x) = e^{x^3+3x^2}$ over $[-5, -1]$.

Solution:

   (a) We are given that $a(t) = -12$, $v(0) = 36$, and $s(0) = 42$. Using antidifferentiation and the given initial conditions, we find that $v(t) = -12t + 36$ and $s(t) = -6t^2 + 36t + 42$.
   We now need to find the time when the stone strikes the ground. So, we have

$$0 = s(t) = -6(t-7)(t+1),$$

   which yields a solution of $t = 7$. (We can disregard the negative solution, as this would be before the stone is thrown. That is, it is outside the domain of our functions.)
   So, the velocity of the stone when it strikes the ground is $v(7) = -48$ meters per second.

   (b) We see that $f'(x) = 3x(x+2)e^{x^3+3x^2}$, which exists for all x in $[-5, -1]$. We see that $f'(x) = 0$ when $x = -2, 0$, but $x = 0$ is not in our domain. So, the only critical number is $x = -2$. Checking the critical number and the endpoints, we see

$$\begin{aligned} f(-5) &= e^{-50} \\ f(-2) &= e^{4} \\ f(-1) &= e^{2}. \end{aligned}$$

   So, the absolute maximum is e^4 and the absolute minimum is e^{-50}.

3. (20 pts) Let $y(x) = \left(1 + \frac{1}{x^2}\right)^x$

   (a) Find the value of $y'(1)$.

   (b) Evaluate $\lim_{x\to\infty} y(x)$.

Solution:

   (a) Use logarithmic differentiation.

$$\begin{aligned} y(x) &= \left(1 + \frac{1}{x^2}\right)^x \\ \ln y &= \ln\left[\left(1 + \frac{1}{x^2}\right)^x\right] = x\ln\left(1 + \frac{1}{x^2}\right) \\ \frac{d}{dx}[\ln y] &= \frac{d}{dx}\left[x\ln\left(1 + \frac{1}{x^2}\right)\right] \\ \frac{y'}{y} &= x \cdot \frac{-\frac{2}{x^3}}{1 + \frac{1}{x^2}} + \ln\left(1 + \frac{1}{x^2}\right) = -\frac{\frac{2}{x^2}}{1 + \frac{1}{x^2}} + \ln\left(1 + \frac{1}{x^2}\right) \end{aligned}$$

$$y' = y\left[\ln\left(1+\frac{1}{x^2}\right) - \frac{2}{x^2+1}\right]$$

The term y in the preceding equation is evaluated at $x = 1$ as $y(1) = \left(1+\frac{1}{1^2}\right)^1 = 2$. Therefore,

$$y'(1) = 2\left[\ln\left(1+\frac{1}{1^2}\right) - \frac{2}{1^2+1}\right] = \boxed{2\,(\ln 2 - 1)}$$

Note that an alternative approach would be to recognize that

$$\left(1+\frac{1}{x^2}\right)^x = e^{\ln\left[\left(1+\frac{1}{x^2}\right)^x\right]} = e^{x\ln\left(1+\frac{1}{x^2}\right)}$$

so that

$$\frac{d}{dx}\left[\left(1+\frac{1}{x^2}\right)^x\right] = \frac{d}{dx}\left[e^{x\ln\left(1+\frac{1}{x^2}\right)}\right]$$

$$= e^{x\ln\left(1+\frac{1}{x^2}\right)} \cdot \frac{d}{dx}\left[x\ln\left(1+\frac{1}{x^2}\right)\right]$$

$$= \left(1+\frac{1}{x^2}\right)^x \cdot \frac{d}{dx}\left[x\ln\left(1+\frac{1}{x^2}\right)\right]$$

(b) The limit to be evaluated is an indeterminate form of type 1^∞. Therefore, a reasonable first step is to take the natural logarithm of both sides, as was done in part (a).

$$y(x) = \left(1+\frac{1}{x^2}\right)^x$$

$$\ln y = \ln\left[\left(1+\frac{1}{x^2}\right)^x\right] = x\ln\left(1+\frac{1}{x^2}\right)$$

Next, take the limit as x approaches ∞ of the expressions on each side of the preceding equation.

$$\lim_{x\to\infty} \ln y = \ln\left[\lim_{x\to\infty} y\right] = \lim_{x\to\infty}\left[x\ln\left(1+\frac{1}{x^2}\right)\right]$$

In the preceding equation, we were able to move the limit into the argument of the natural logarithm function because that function is continuous (and the limit is presumed to exist). The limit on the far right is an indeterminate form of type $0\cdot\infty$ (actually, $\infty\cdot 0$ the way it's expressed here). The next step is to express the limit as an indeterminate form of type $0/0$ so that L'Hôpital's Rule can be applied.

$$\ln\left[\lim_{x\to\infty} y\right] = \lim_{x\to\infty}\left[\frac{\ln\left(1+\frac{1}{x^2}\right)}{\frac{1}{x}}\right] \overset{\text{LH}}{=} \lim_{x\to\infty}\left[\frac{\left(-\frac{2}{x^3}\right)/\left(1+\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)}\right]$$

$$= \lim_{x\to\infty}\left(\frac{2}{x}\right)\left(\frac{1}{1+\frac{1}{x^2}}\right) = 0$$

$$\ln\left[\lim_{x\to\infty} y\right] = 0 \quad\Rightarrow\quad \lim_{x\to\infty} y = e^0 = \boxed{1}$$

Note that an alternative approach would be to recognize that

$$\left(1+\frac{1}{x^2}\right)^x = e^{\ln\left[\left(1+\frac{1}{x^2}\right)^x\right]} = e^{x\ln\left(1+\frac{1}{x^2}\right)}$$

so that

$$\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x = \lim_{x\to\infty}\left[e^{x\ln\left(1+\frac{1}{x^2}\right)}\right] = e^{\lim_{x\to\infty}\left[x\ln\left(1+\frac{1}{x^2}\right)\right]} \quad ,$$

where the limit was moved into the argument of the natural exponential function because that function is continuous (and the limit is presumed to exist).

4. (28 pts) The following problems are not related.

 (a) Find the derivative of $y = x^2\arctan(e^{5x})$.

 (b) Evaluate $\displaystyle\int_0^{\frac{\sqrt{3}}{4}} \frac{1}{\sqrt{1-4x^2}}\,dx$.

 (c) Evaluate $\displaystyle\int_1^2 \frac{\sinh(\ln x)}{x}\,dx$.

Solution:

 (a) We use the product rule to get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(x^2\arctan(e^{5x})\right)\\ &= \frac{5x^2e^{5x}}{1+e^{10x}} + 2x\arctan\left(e^{5x}\right).\end{aligned}$$

 (b) If we let $u = 2x$, then we have $du = 2dx$ and new limits of integration $u = \frac{\sqrt{3}}{2}$ and $u = 0$.

$$\begin{aligned}\int_0^{\frac{\sqrt{3}}{4}} \frac{1}{\sqrt{1-4x^2}}\,dx &= \int_0^{\frac{\sqrt{3}}{4}} \frac{1}{\sqrt{1-(2x)^2}}\,dx\\ &= \frac{1}{2}\int_0^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-u^2}}\,du\\ &= \frac{1}{2}\left(\arcsin\left(\frac{\sqrt{3}}{2}\right) - \arcsin(0)\right)\\ &= \frac{\pi}{6}.\end{aligned}$$

 (c)

$$\begin{aligned}\int_1^2 \frac{\sinh(\ln x)}{x}\,dx &= \int_1^2 \frac{e^{\ln x}-e^{-\ln x}}{2x}\,dx\\ &= \int_1^2 \frac{1}{2}\left(1-\frac{1}{x^2}\right)\,dx\end{aligned}$$

$$= \frac{1}{2}\left[x + \frac{1}{x}\right]_1^2$$
$$= \frac{1}{4}.$$

OR:

If we let $u = \ln x$, then $du = \frac{1}{x}\,dx$, and the new limits of integration are $u = \ln 2$ and $u = \ln 1 = 0$.

$$\begin{aligned}\int_1^2 \frac{\sinh(\ln x)}{x}\,dx &= \int_0^{\ln 2} \sinh(u)\,du \\ &= \cosh(\ln 2) - \cosh(0) \\ &= \frac{e^{\ln 2} + e^{-\ln 2}}{2} - \frac{e^0 + e^{-0}}{2} \\ &= \frac{1}{4}.\end{aligned}$$

5. (24 pts) The following problems are not related.

(a) Solve the following inequality: $\log_{10}(4 - 3x) < 1$. State your final answer using interval notation.

(b) Suppose that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$. Find $(f/g)'(5)$.

(c) Evaluate $\lim_{x\to 0} \frac{1}{x}\int_0^x \sin(t^2+1)\,dt$

Solution:

(a) $\log_{10}(4 - 3x) < 1$ implies

$$0 < 4 - 3x < 10^1$$

$$-4 < -3x < 6$$

$$-2 < x < \frac{4}{3}$$

So, the solution is $\left(-2, \frac{4}{3}\right)$.

(b) Suppose that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$. Find $(f/g)'(5)$.

$$(f/g)'(5) = \frac{g(5)f'(5) - f(5)g'(5)}{[g(5)]^2} = -\frac{20}{9}$$

(c) Note that

$$\lim_{x\to 0} \frac{1}{x}\int_0^x \sin(t^2+1)\,dt = \lim_{x\to 0} \frac{\int_0^x \sin(t^2+1)\,dt}{x}$$

is a $\frac{0}{0}$ indeterminate form. So, we can apply L'Hospital's Rule. When we do, the limit becomes

$$\lim_{x\to 0} \frac{\sin(x^2+1)}{1} = \sin(1).$$

6. (16 pts) A rectangle is growing, but its length is always twice its width. Initially, the width of the rectangle is 3 centimeters. Two minutes later the width is 5 centimeters. If the rate of change of the width is proportional to the width (that is, $\frac{dw}{dt} = kw$), find the rate of change of the area of the rectangle after ten minutes.

Solution:

Let $w(t)$ be the width of the rectangle (in centimeters) after t minutes. We know that $w(t) = w_0 e^{kt}$ where $w_0 = w(0) = 3$ and $w(2) = 5$. We can use the latter point to find that $k = \ln\sqrt{\frac{5}{3}}$. Thus, $w(t) = 3e^{\sqrt{\frac{5}{3}}t}$. Further, the area of the rectangle is given by

$$A(t) = (2w(t))(w(t)) = 18e^{\ln(\frac{5}{3})t}.$$

So, we have

$$A'(10) = 18 \cdot \ln\left(\frac{5}{3}\right) e^{\ln(\frac{5}{3})10} \text{ centimeters squared per minute.}$$

7. (16 pts) Sketch a graph of a single function $y = f(x)$ with all of the following properties:

- The domain of f is $(-3, 3)$.
- $\lim_{x\to a} f(x) = f(a)$ for all a except $a = 0$
- $f(1) = 0$
- $f(-2) = 0$
- $\lim_{x\to 3^-} f(x) = \infty$
- $\lim_{x\to 0^-} f(x) = -\infty$
- $f''(x) > 0$ for x in $(0, 3)$
- f is an odd function

Solution: