1. (40 pts) The following problems are not related.

(a) Evaluate \( \int_1^9 \frac{r + 3}{\sqrt{r}} \, dr \).

(b) Evaluate \( \int_0^{\pi/2} \tan^5 x \sec^2 x \, dx \).

(c) Evaluate \( \int_0^2 [x^3 - 2f(x)] \, dx \) where \( \int_0^2 f(x) \, dx = 6 \).

(d) Evaluate the sum: \( \sum_{i=1}^{40} 5(i - 1)^2 \). (You do not need to simplify your final answer, but it should be in a form that could be directly input into a calculator.)

(e) If the average value of \( h \) on \([-2, 6]\) is 4, then evaluate \( \int_{-2}^6 h(x) \, dx \).

Solution:

(a)

\[
\int_1^9 \frac{r + 3}{\sqrt{r}} \, dr = \int_1^9 \left[ \frac{1}{2}r^{\frac{1}{2}} + 3r^{-\frac{1}{2}} \right] \, dr \\
= \left[ \frac{2}{3}r^{\frac{3}{2}} + 6r^{\frac{1}{2}} \right]_1^9 \\
= [18 + 18] - \left[ \frac{2}{3} + 6 \right] \\
= \frac{88}{3}.
\]

(b) If we let \( u = \tan x \), then \( du = \sec^2 x \, dx \), and the new limits of integration become \( u = \tan \frac{\pi}{4} = 1 \) and \( u = \tan 0 = 0 \):

\[
\int_0^{\pi/2} \tan^5 x \sec^2 x \, dx = \int_0^1 u^5 \, du \\
= \left[ \frac{1}{6}u^6 \right]_0^1 \\
= \frac{1}{6}.
\]

(c)

\[
\int_0^2 [x^3 - 2f(x)] \, dx = \int_0^2 x^3 \, dx - 2 \int_0^2 f(x) \, dx \\
= \left[ \frac{1}{4}x^4 \right]_0^2 - 2(6) \\
= -8.
\]
(d) 
\[
\sum_{i=1}^{40} 5(i - 1)^2 = 5 \sum_{i=1}^{40} (i^2 - 2i + 1) \\
= 5 \left[ \sum_{i=1}^{40} i^2 - 2 \sum_{i=1}^{40} i + \sum_{i=1}^{40} 1 \right] \\
= 5 \left( \frac{40(41)(81)}{6} \right) - 2 \left( \frac{40(41)}{2} + 40 \right).
\]

(e) We are given 
\[
\text{Avg}(h) = \frac{1}{6 - (-2)} \int_{-2}^{6} h(x) \, dx = 4.
\]

Therefore, 
\[
\int_{-2}^{6} h(x) \, dx = 32.
\]

2. (24 pts) Consider the function \( f(x) = x - \cos x \).

(a) Estimate the location of the \( x \)-intercept of \( f(x) \) by applying one iteration of Newton’s method with an initial approximation of \( x_0 = \pi/6 \). Fully simplify your result.

(b) Use the Right Endpoint Rule with \( n = 3 \) to approximate the value of \( \int_{0}^{\pi} f(x) \, dx \).

(c) Find the derivative with respect to \( x \) of \( g(x) = \int_{x^2+1}^{0} f(t) \, dt \).

Solution:

(a) 
\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n - \cos (x_n)}{1 + \sin (x_n)} \quad x_0 = \frac{\pi}{6},
\]

\[
x_1 = \frac{\pi}{6} - \frac{\left[ \frac{\pi}{6} - \cos \left( \frac{\pi}{6} \right) \right]}{1 + \sin \left( \frac{\pi}{6} \right)} = \frac{\pi}{6} - \frac{\left[ \frac{\pi}{6} - \frac{\sqrt{3}}{2} \right]}{1 + \frac{1}{2}} = \frac{\pi}{6} \frac{3}{2} = \frac{\pi + 6\sqrt{3}}{18}.
\]

(b) 
\[
\int_{0}^{\pi} (x - \cos x) \, dx \approx \frac{\pi}{3} \left[ f \left( \frac{\pi}{3} \right) + f \left( \frac{2\pi}{3} \right) + f (\pi) \right] \\
= \frac{\pi}{3} \left[ \frac{\pi}{3} - \cos \left( \frac{\pi}{3} \right) + \frac{2\pi}{3} - \cos \left( \frac{2\pi}{3} \right) + \pi - \cos (\pi) \right]
\]
\[
\pi \left[ 2\pi - \frac{1}{2} - \left( -\frac{1}{2} \right) - (-1) \right] = \frac{\pi (2\pi + 1)}{3}
\]

(c) Apply Part (1) of the Fundamental Theorem of Calculus.

\[
\frac{d}{dx} \int_0^{x^2 + 1} (t - \cos t) dt = -\frac{d}{dx} \int_0^{x^2 + 1} (t - \cos t) dt = -\left[ (x^2 + 1) - \cos (x^2 + 1) \right] (2x) \]

Alternate solution:

\[
\int_0^{x^2 + 1} (t - \cos t) dt = \left[ \frac{t^2}{2} - \sin t \right]_0^{x^2 + 1} = -\frac{(x^2 + 1)^2}{2} + \sin (x^2 + 1)
\]

3. (14 pts) Suppose that you want to build a cylindrical water tank with \(9\pi \text{ ft}^3\) capacity. The cost of building each square foot of wall is $1, each square foot of the bottom base costs $6 and building each square foot of the top costs $3. Find the dimensions that will minimize the cost of building the tank.

**Solution:** Let \(r\) be the radius of the cylindrical tank and let \(h\) be its height. We’re given that the volume \(V = \pi r^2 h = 9\pi\). Thus, \(h = 9/r^2\). We want to find the dimensions that will minimize the cost of the tank.

\[
\text{Cost} = \text{cost of the side} + \text{cost of the top} + \text{cost of the bottom}
\]

\[
= 2\pi rh + 3\pi r^2 + 6\pi r^2
\]

\[
= 2\pi rh + 9\pi r^2
\]

Substitute in \(h = 9/r^2\) to obtain the cost as a function of \(r\)

\[
C(r) = \frac{18\pi}{r} + 9\pi r^2
\]

We note that the domain of \(C(r)\) is \((0, \infty)\). Differentiate \(C(r)\) to find the critical points:

\[
C'(r) = -\frac{18\pi}{r^2} + 18\pi r
\]

Set \(C'(r) = 0\) to obtain the critical point \(r = 1\).

To establish that \(r = 1\) is an absolute minimum on the domain \((0, \infty)\) we can use the first derivative test to find that \(C\) is decreasing on \((0, 1)\) and increasing on \((1, \infty)\) or we can use the second derivative test to find

\[
C''(r) = \frac{36\pi}{r^3} + 18\pi
\]

and \(C''(r) > 0\) for all \(r > 0\). Thus, \(C\) is concave up on the interval \((0, \infty)\), which implies that the absolute minimum is located at \(r = 1\).

Answer: \(r=1\) ft and \(h=9\) ft give the dimensions that minimize the cost.
4. (22 pts) Let \( g(x) = \frac{x^2 - 9}{x^2 + 9} \) and define \( A(x) = \int_{-7}^{x} g(t) \, dt \).

(a) On which interval(s) is \( A \) increasing? Decreasing?
(b) On which interval(s) is \( A \) concave up? Concave down?
(c) Draw a graph of \( A(x) \) for \(-7 \leq x \leq 7\) that clearly shows the \( x \)-coordinates of local extrema, inflection points, and intercepts. Assume \( \int_{-7}^{0} g(x) \, dx = 0 \) to determine the intercepts. (This is a close approximation, but you may assume it is exact for the purposes of your graph.)

Solution:
(a) To begin with, \( A'(x) = \frac{x^2 - 9}{x^2 + 9} \). The critical points occur at \( x = \pm 3 \). On the interval \((-\infty, -3)\) we can check \( A'(-5) = \frac{16}{34} > 0 \), so \( A \) is increasing. On the interval \((-3, 3)\) we can check \( A'(0) = -1 < 0 \), so \( A \) is decreasing. On the interval \((3, \infty)\) we can check \( A'(5) = \frac{16}{34} > 0 \), so \( A \) is increasing.

(b)
\[
A''(x) = \frac{d}{dx} \left[ \frac{x^2 - 9}{x^2 + 9} \right] = \frac{2x(x^2 + 9) - 2x(x^2 - 9)}{(x^2 + 9)^2} = \frac{36x}{(x^2 + 9)^2}
\]

The only inflection point is at \( x = 0 \). On the interval \((-\infty, 0)\) we can check \( A''(-1) = -3.6 < 0 \), so \( A \) is concave down. On the interval \((0, \infty)\) we can check \( A''(1) = 3.6 > 0 \), so \( A \) is concave up.

(c) Beyond what we’ve already determined about the function from the first and second derivative, we also know
\[
A(-7) = \int_{-7}^{-7} g(x) \, dx = 0
\]
and
\[
A(0) = \int_{-7}^{0} g(x) \, dx = 0.
\]

Because \( g(x) \) is even and \( \int_{-7}^{0} g(x) \, dx = 0 \), we also know that
\[
A(7) = \int_{-7}^{7} g(x) \, dx = 2 \int_{-7}^{0} g(x) \, dx = 0.
\]

All of this results in the figure