1. (40 pts) The following problems are not related.

   (a) Evaluate $\displaystyle\int_1^9 \frac{r+3}{\sqrt{r}}\,dr$.

   (b) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \tan^5 x \sec^2 x\,dx$.

   (c) Evaluate $\displaystyle\int_0^2 [x^3 - 2f(x)]\,dx$ where $\displaystyle\int_0^2 f(x)\,dx = 6$.

   (d) Evaluate the sum: $\displaystyle\sum_{i=1}^{40} 5(i-1)^2$. (You do not need to simplify your final answer, but it should be in a form that could be directly input into a calculator.)

   (e) If the average value of h on $[-2,6]$ is 4, then evaluate $\displaystyle\int_{-2}^{6} h(x)\,dx$.

Solution:

(a)

$$\begin{aligned}\int_1^9 \frac{r+3}{\sqrt{r}}\,dr &= \int_1^9 \left[r^{\frac{1}{2}} + 3r^{-\frac{1}{2}}\right]\,dr\\ &= \left[\frac{2}{3}r^{\frac{3}{2}} + 6r^{\frac{1}{2}}\right]_1^9\\ &= [18+18] - [\frac{2}{3}+6]\\ &= \boxed{\frac{88}{3}}.\end{aligned}$$

(b) If we let $u = \tan x$, then $du = \sec^2\,dx$, and the new limits of integration become $u = \tan\frac{\pi}{4} = 1$ and $u = \tan 0 = 0$:

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \tan^5 x \sec^2 x\,dx &= \int_0^1 u^5\,du\\ &= \left[\frac{1}{6}u^6\right]_0^1\\ &= \boxed{\frac{1}{6}}.\end{aligned}$$

(c)

$$\begin{aligned}\int_0^2 [x^3 - 2f(x)]\,dx &= \int_0^2 x^3\,dx - 2\int_0^2 f(x)\,dx\\ &= \left[\frac{1}{4}x^4\right]_0^2 - 2(6)\\ &= \boxed{-8}.\end{aligned}$$

(d)

$$\begin{aligned}\sum_{i=1}^{40} 5(i-1)^2 &= 5\sum_{i=1}^{40}(i^2-2i+1)\\ &= 5\left[\sum_{i=1}^{40} i^2 - 2\sum_{i=1}^{40} i + \sum_{i=1}^{40} 1\right]\\ &= \boxed{5\left(\left(\frac{40(41)(81)}{6}\right) - 2\left(\frac{40(41)}{2}\right) + 40\right)}.\end{aligned}$$

(e) We are given

$$\text{Avg}(h) = \frac{1}{6-(-2)}\int_{-2}^{6} h(x)\ dx = 4$$

Therefore,

$$\boxed{\int_{-2}^{6} h(x)\ dx = 32}$$

2. (24 pts) Consider the function $f(x) = x - \cos x$.

(a) Estimate the location of the x-intercept of $f(x)$ by applying one iteration of Newton's method with an initial approximation of $x_0 = \pi/6$. Fully simplify your result.

(b) Use the Right Endpoint Rule with $n = 3$ to approximate the value of $\displaystyle\int_0^{\pi} f(x)\,dx$.

(c) Find the derivative with respect to x of $g(x) = \displaystyle\int_{x^2+1}^{0} f(t)\,dt$.

Solution:

(a)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n - \cos(x_n)}{1+\sin(x_n)}\ , \quad x_0 = \frac{\pi}{6}$$

$$x_1 = \frac{\pi}{6} - \frac{\left[\frac{\pi}{6} - \cos\left(\frac{\pi}{6}\right)\right]}{\left[1+\sin\left(\frac{\pi}{6}\right)\right]} = \frac{\pi}{6} - \frac{\left[\frac{\pi}{6} - \frac{\sqrt{3}}{2}\right]}{\left[1+\frac{1}{2}\right]}$$

$$= \frac{\pi}{6} - \frac{2}{3}\left[\frac{\pi}{6} - \frac{\sqrt{3}}{2}\right] = \boxed{\frac{\pi}{18} + \frac{\sqrt{3}}{3}} = \boxed{\frac{\pi + 6\sqrt{3}}{18}}$$

(b)

$$\begin{aligned}\int_0^{\pi}(x-\cos x)\,dx &\approx \frac{\pi}{3}\left[f\left(\frac{\pi}{3}\right) + f\left(\frac{2\pi}{3}\right) + f(\pi)\right]\\ &= \frac{\pi}{3}\left[\frac{\pi}{3} - \cos\left(\frac{\pi}{3}\right) + \frac{2\pi}{3} - \cos\left(\frac{2\pi}{3}\right) + \pi - \cos(\pi)\right]\end{aligned}$$

$$= \frac{\pi}{3}\left[2\pi - \frac{1}{2} - \left(-\frac{1}{2}\right) - (-1)\right] = \boxed{\frac{\pi(2\pi+1)}{3}}$$

(c) Apply Part (1) of the Fundamental Theorem of Calculus.

$$\frac{d}{dx}\int_{x^2+1}^{0}(t-\cos t)dt = -\frac{d}{dx}\int_{0}^{x^2+1}(t-\cos t)dt = -\left[\left(x^2+1\right) - \cos\left(x^2+1\right)\right](2x)$$

$$= \boxed{-2x\left[\left(x^2+1\right) - \cos\left(x^2+1\right)\right]}$$

Alternate solution:

$$\int_{x^2+1}^{0}(t-\cos t)dt = \left[\frac{t^2}{2} - \sin t\right]\Bigg|_{x^2+1}^{0} = -\frac{\left(x^2+1\right)^2}{2} + \sin\left(x^2+1\right)$$

$$\frac{d}{dx}\left[-\frac{\left(x^2+1\right)^2}{2} + \sin\left(x^2+1\right)\right] = -\left(x^2+1\right)(2x) + \cos\left(x^2+1\right)(2x)$$

$$= \boxed{-2x\left[\left(x^2+1\right) - \cos\left(x^2+1\right)\right]}$$

3. (14 pts) Suppose that you want to build a cylindrical water tank with 9π ft^3 capacity. The cost of building each square foot of wall is \$1, each square foot of the bottom base costs \$6 and building each square foot of the top costs \$3. Find the dimensions that will minimize the cost of building the tank.

Solution: Let r be the radius of the cylindrical tank and let h be its height. We're given that the volume $V = \pi r^2 h = 9\pi$. Thus, $h = 9/r^2$. We want to find the dimensions that will minimize the cost of the tank.

$$\begin{aligned}\text{Cost} &= \text{cost of the side} + \text{cost of the top} + \text{cost of the bottom}\\ &= 2\pi r h + 3\pi r^2 + 6\pi r^2\\ &= 2\pi r h + 9\pi r^2\end{aligned}$$

Substitute in $h = 9/r^2$ to obtain the cost as a function of r

$$C(r) = \frac{18\pi}{r} + 9\pi r^2$$

We note that the domain of $C(r)$ is $(0, \infty)$. Differentiate $C(r)$ to find the critical points:

$$C'(r) = \frac{-18\pi}{r^2} + 18\pi r$$

Set $C'(r) = 0$ to obtain the critical point $r = 1$.

To establish that $r = 1$ is an absolute minimum on the domain $(0, \infty)$ we can use the first derivative test to find that C is decreasing on $(0, 1)$ and increasing on $(1, \infty)$ or we can use the second derivative test to find

$$C''(r) = \frac{36\pi}{r^3} + 18\pi$$

and $C''(r) > 0$ for all $r > 0$. Thus, C is concave up on the interval $(0, \infty)$, which implies that the absolute minimum is located at $r = 1$.

Answer: $\boxed{\text{r=1 ft and h=9 ft give the dimensions that minimize the cost.}}$

4. (22 pts) Let $g(x) = \dfrac{x^2-9}{x^2+9}$ and define $A(x) = \displaystyle\int_{-7}^{x} g(t)\,dt$.

(a) On which interval(s) is A increasing? Decreasing?

(b) On which interval(s) is A concave up? Concave down?

(c) Draw a graph of $A(x)$ for $-7 \le x \le 7$ that clearly shows the x-coordinates of local extrema, inflection points, and intercepts. Assume $\displaystyle\int_{-7}^{0} g(x)\,dx = 0$ to determine the intercepts. (This is a close approximation, but you may assume it is exact for the purposes of your graph.)

Solution:

(a) To begin with, $A'(x) = \dfrac{x^2-9}{x^2+9}$. The critical points occur at $x = \pm 3$. On the interval $(-\infty, -3)$ we can check $A'(-5) = \frac{16}{34} > 0$, so A is increasing. On the interval $(-3, 3)$ we can check $A'(0) = -1 < 0$, so A is decreasing. On the interval $(3, \infty)$ we can check $A'(5) = \frac{16}{34} > 0$, so A is increasing.

(b)

$$\begin{aligned} A''(x) &= \frac{d}{dx}\left[\frac{x^2-9}{x^2+9}\right] \\ &= \frac{2x(x^2+9) - 2x(x^2-9)}{(x^2+9)^2} \\ &= \frac{36x}{(x^2+9)^2} \end{aligned}$$

The only inflection point is at $x = 0$. On the interval $(-\infty, 0)$ we can check $A''(-1) = -3.6 < 0$, so A is concave down. On the interval $(0, \infty)$ we can check $A''(1) = 3.6 > 0$, so A is concave up.

(c) Beyond what we've already determined about the function from the first and second derivative, we also know

$$A(-7) = \int_{-7}^{-7} g(x)\,dx = 0$$

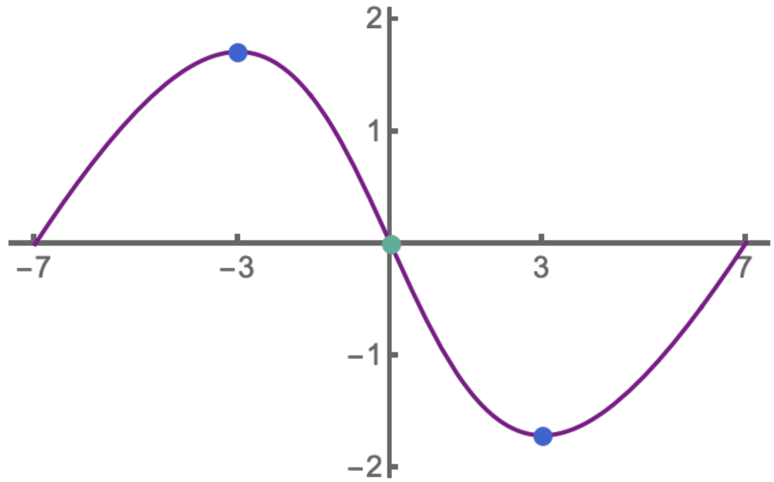
and

$$A(0) = \int_{-7}^{0} g(x)\,dx = 0.$$

Because $g(x)$ is even and $\displaystyle\int_{-7}^{0} g(x)\,dx = 0$, we also know that

$$A(7) = \int_{-7}^{7} g(x)\,dx = 2\int_{-7}^{0} g(x)\,dx = 0.$$

All of this results in the figure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