1. (30 pts) The following problems are not related.

   (a) Find the derivative of $f(x) = \dfrac{\sec(x^2)}{ax+7}$ where a is a constant. Do **Not** simplify your final answer.

   (b) Find the equation of the tangent line of $x^2 + xy - y^3 = 7$ at the point $(-2, -1)$.

   (c) Determine the (x, y)-coordinates where $y = x^2\sqrt{2-x}$ has horizontal tangent lines.

   Solution:

   (a) Using the quotient and chain rules, we see that

   $$\boxed{f'(x) = \frac{(ax+7)(2x)\sec(x^2)\tan(x^2) - a\sec(x^2)}{(ax+7)^2}.}$$

   (b) Begin by differentiating both sides of $x^2 + xy - y^3 = 7$ with respect to x we get

   $$2x + x\frac{dy}{dx} + y - 3y^2\frac{dy}{dx} = 0.$$

   Solving for $\frac{dy}{dx}$, we get

   $$\frac{dy}{dx} = \frac{2x+y}{3y^2-x}.$$

   At $(-2, -1)$ we have $\frac{dy}{dx} = -1$. The line with this slope through $(-2, -1)$ is $\boxed{y = -x - 3.}$

   (c) We differentiate to see that

   $$\begin{aligned}\frac{dy}{dx} &= x^2\left(\frac{-1}{2\sqrt{2-x}}\right) + 2x\sqrt{2-x} \\ &= \frac{-5x^2+8x}{2\sqrt{2-x}}.\end{aligned}$$

   If we set $\frac{dy}{dx} = 0$ and solve for x, we see that $x = 0, \frac{8}{5}$ are the solutions. So, the two points on this curve with horizontal tangents are $\boxed{(0,0)}$ and $\boxed{\left(\frac{8}{5}, \frac{64}{25}\sqrt{\frac{2}{5}}\right)}$.

2. (35 pts) The following problems are not related.

   (a) Consider

   $$g(x) = \begin{cases} \sqrt{x} + 2 & 0 \le x \le 4 \\ x & x > 4 \end{cases}.$$

   Use the limit definition of the derivative to explain why the function g does not have a derivative when $x = 4$. (Explanations that do not use the limit definition of the derivative will earn no credit here.)

(b) Consider a runner whose position function is $s(t) = t^2 + 5t + 50$ feet. Suppose a measurement of time $t = 10$ seconds is taken, with a possible error in measurement of up to 0.1 seconds. If the measured value is subsequently used to calculate the value of $y = s(10)$, use differentials to estimate the corresponding absolute error and relative error in the calculated y value.

(c) Verify that the function $f(x) = x^3 - 6x$ satisfies the hypotheses of the Mean Value Theorem on the interval $[0, 3]$, and find all numbers c that satisfy the conclusion of that theorem.

Solution:

(a) The function g is differentiable at $x = 4$ if

$$\lim_{h\to 0} \frac{g(4+h) - g(4)}{h}$$

exists. Calculate:

$$\lim_{h\to 0^+} \frac{g(4+h) - g(4)}{h} = \lim_{h\to 0+} \frac{4+h-4}{h} = 1$$

and

$$\begin{aligned}
\lim_{h\to 0^-} \frac{g(4+h) - g(4)}{h} &= \lim_{h\to 0^-} \frac{\sqrt{4+h}+2-4}{h} \\
&= \lim_{h\to 0^-} \frac{(\sqrt{4+h}-2)}{h} \frac{(\sqrt{4+h}+2)}{(\sqrt{4+h}+2)} \\
&= \lim_{h\to 0^-} \frac{h}{h(\sqrt{4+h}+2)} \\
&= \frac{1}{4}
\end{aligned}$$

Since $\lim_{h\to 0^+} \frac{g(4+h) - g(4)}{h} \neq \lim_{h\to 0^-} \frac{g(4+h) - g(4)}{h}$ we conclude that $\lim_{h\to 0} \frac{g(4+h) - g(4)}{h}$ does not exist so g is not differentiable at $x = 4$.

(b) The differential dy is defined to be $dy = s'(t)dt$.

$$dy = s'(t)dt = (2t+5)dt$$

In this problem, we have $t = 10$ and $dt = 0.1$.

$$dy|_{t=10} = s'(10)dt = (25)(0.1) = 2.5$$

Therefore, the absolute error is $\boxed{2.5 \text{ ft}}$

The nominal position of the runner at $t = 10$ seconds is $s(10) = 10^2 + (5)(10) + 50 = 200$ ft.

$$\left.\frac{dy}{y}\right|_{t=10} = \frac{2.5}{200} = \frac{1}{80}$$

Therefore, the relative error is $\boxed{\frac{1}{80} = 0.0125 = 1.25\%}$

(c) The function $f(x) = x^3 - 6x$ is $\boxed{\text{continuous on [0,3]}}$ and $\boxed{\text{differentiable on (0,3)}}$ so that $f(x)$ satisfies both hypotheses of the Mean Value Theorem.

The Mean Value Theorem asserts that there exists at least one number c on the interval $(0, 3)$ such that

$$f'(c) = \frac{f(3) - f(0)}{3 - 0}$$

The derivative of $f(x)$ is $f'(x) = 3x^2 - 6$ Also, $f(0) = 0$ and $f(3) = 27 - 18 = 9$. Therefore, there exists at least one number c on the interval $(0, 3)$ such that

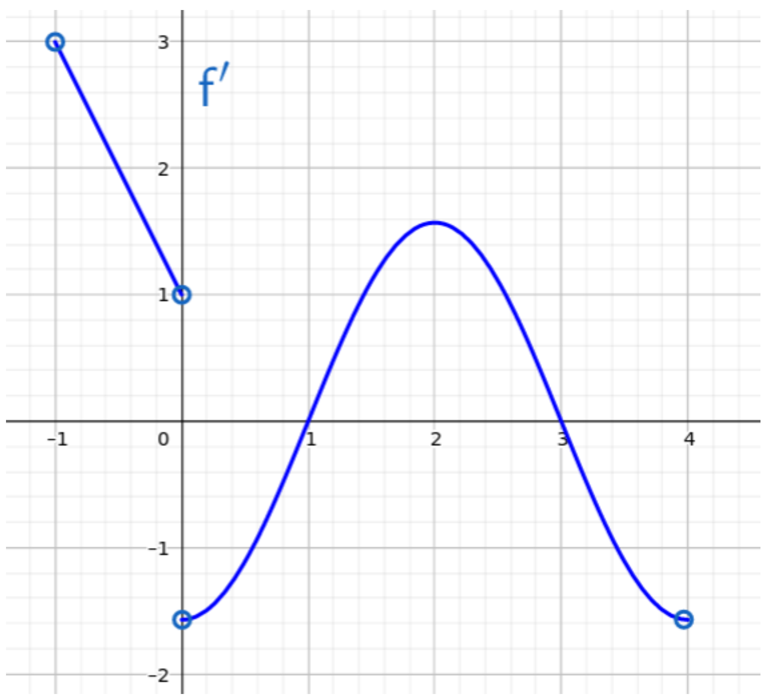
$$3c^2 - 6 = \frac{9 - 0}{3} = 3$$

Solving the preceding equation leads to the following candidates for c:

$$c^2 = 3, \Rightarrow c = \pm\sqrt{3}$$

Although the number $\sqrt{3}$ lies on the interval $(0, 3)$, the number $-\sqrt{3}$ does not. Therefore, the only number c that satisfies the conclusion of the Mean Value Theorem is $\boxed{c = \sqrt{3}}$

3. (20 pts) f is a continuous function on $[-1, 4]$ and the following graph corresponds to f'. Answer the following questions. No justification is required.



(a) On which interval(s) is f increasing?

(b) For what x-values in $(-1, 4)$ is f not differentiable?

(c) For what x-values in $(-1, 4)$ does f have a local maximum? For what x-values in $(-1, 4)$ does f have a local minimum?

(d) On what interval(s) is f concave down?

(e) Assume $f(0) = 0$. Sketch a graph of a function f that corresponds to this graph of $y = f'(x)$ and your answers from above. Clearly label the x-coordinates of all local maximums and minimums and inflection points. (There are many correct solutions.)

Solution:

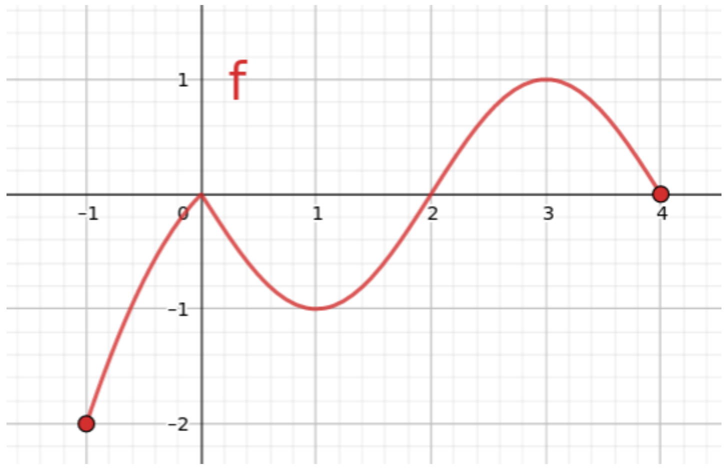
(a) f is increasing on $(-1, 0) \cup (1, 3)$.

(b) f is not differentiable on $x = 0$.

(c) local maximums at $x = 0$ and $x = 3$, local minimum at $x = 1$.

(d) f is concave down on $(-1, 0)$ and $(2, 4)$

(e)



4. (15 pts)

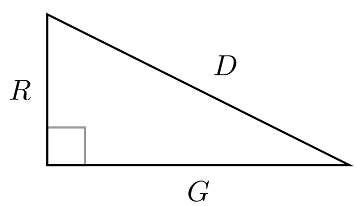
Rosencrantz and Guildenstern were riding together on their way visit an old friend when the road diverged into two paths. Rosencrantz had a bike and took a road going North at 8 mph. Guildenstern, on the other hand, had a motorcycle and took a road going East at 15 mph.

(a) Draw a picture illustrating the situation taking care to label all variables. Write down an equation(s) that relates all the variables.

(b) How fast is the distance between them changing after an hour?

Solution:

(a)



We see that the distance Rosencrantz has traveled (R) and the distance Guildenstern has traveled (G) form a right triangle with the distance between the two travelers (D) forming the hypotenuse. This means the equation relating all three variables is

$$R^2 + G^2 = D^2. \tag{1}$$

(b) To calculate how fast the distance between them is changing, we first need their current positions. After one hour $R = 8$ miles, $G = 15$ miles, and using equation (1) we find $D = 17$. We now take the derivative of equation (1) with respect to time to arrive at

$$2\frac{dR}{dt}R + 2\frac{dG}{dt}G = 2\frac{dD}{dt}D \tag{2}$$

From the problem set up, know $\frac{dR}{dt} = 8$ and $\frac{dG}{dt} = 15$. Plugging all of this into (1) leaves us with

$$2 \cdot 8 \cdot 8 + 2 \cdot 15 \cdot 15 = 2\frac{dD}{dt} \cdot 17.$$

Solving this gives $\frac{dD}{dt} = 17$.