1. (24 pts) Trigonometry

   (a) A plane takes off in a straight line with an angle of inclination of $\pi/3$. How far (in the horizontal direction) will the plane have traveled when it reaches a height of 12 km?

   (b) Let $f(\theta) = 6\cos(2\theta) - 1$

   i. Sketch a graph of f on the interval $[0, 2\pi]$. (On your graph, label the coordinates of the y intercept and the coordinates of the maximum and minimum values of the function.)

   ii. What is the range of f?

   iii. Solve the inequality $6\cos(2\theta) - 1 > 2$. for θ in the interval $[0, 2\pi]$. Write your answer in interval notation.

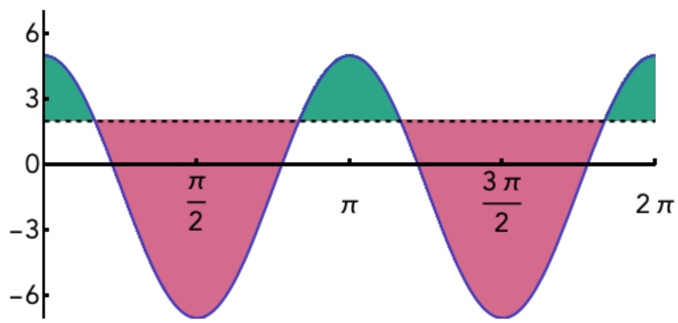
Solution:

(a)

$$\tan\left(\frac{\pi}{3}\right) = \frac{\text{Height of the Plane}}{\text{Horizontal Distance Traveled}}$$
$$\sqrt{3} = \frac{12 \text{ km}}{\text{Horizontal Distance Traveled}}$$
$$4\sqrt{3} \text{ km} = \text{Horizontal Distance Traveled}$$

(b) $f(\theta) = 6\cos(2\theta) - 1$ has period π

i.



ii. The range of f is $[-7, 5]$

iii. Solving $6\cos(2\theta) - 1 > 2$ is equivalent to solving $\cos(2\theta) > 1/2$. We note that for x in the interval $[0, 2\pi]$, $\cos x = 1/2$ when $x = \pi/3$ or $x = 5\pi/3$. Therefore, $\cos(2\theta) = 1/2$ when $2\theta = \pi/3$ or $2\theta = 5\pi/3$. Hence, $\theta = \pi/6$ or $\theta = 5\pi/6$. Since we want to solve the inequality on the interval $[0, 2\pi]$, we use periodicity to obtain the solution $\boxed{[0, \pi/6) \cup (5\pi/6, 7\pi/6) \cup (11\pi/6, 2\pi]}$

2. (26 pts) **Limits** Evaluate the following limits and simplify your answers.
(Reminder: You may not use L'Hopital's Rule in your solution.)

   (a) $\displaystyle\lim_{x\to 3} \frac{|x-3|}{2x^2 - 5x - 3}$

   (b) $\displaystyle\lim_{x\to 0} \frac{\tan(3x)\cos(4x)}{x}$

(c) $\lim_{x \to -\infty} \left(\sqrt{x^2+3x} - \sqrt{x^2-3x} \right)$

Solution:

(a) To deal with the absolute value, we break this problem into a left limit and a right limit.

$$\begin{aligned}\lim_{x\to 3^-} \frac{|x-3|}{2x^2-5x-3} &= \lim_{x\to 3^-} \frac{-(x-3)}{2x^2-5x-3} \\ &= \lim_{x\to 3^-} \frac{-(x-3)}{(2x+1)(x-3)} \\ &= \lim_{x\to 3^-} \frac{-1}{2x+1} \\ &= -\frac{1}{7}\end{aligned}$$

$$\begin{aligned}\lim_{x\to 3^+} \frac{|x-3|}{2x^2-5x-3} &= \lim_{x\to 3^+} \frac{(x-3)}{2x^2-5x-3} \\ &= \lim_{x\to 3^+} \frac{(x-3)}{(2x+1)(x-3)} \\ &= \lim_{x\to 3^+} \frac{1}{2x+1} \\ &= \frac{1}{7}\end{aligned}$$

Since the left limit is not the same as the right limit, we conclude that the limit does not exist.

(b) When approaching a limit that involves trig functions, we aim to isolate one of the special trig limits that we already understand. In this case, $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.

$$\begin{aligned}\lim_{x\to 0} \frac{\tan(3x)\cos(4x)}{x} &= \left(\lim_{x\to 0} \frac{\sin(3x)}{x}\right)\left(\lim_{x\to 0} \frac{\cos(4x)}{\cos(3x)}\right) \\ &= 3\left(\lim_{x\to 0} \frac{\sin(3x)}{3x}\right)(1) \\ &= 3\,(1)\,(1) \\ &= 3\end{aligned}$$

(c) For a limit that involves the difference of square roots (or indeed any radicals), we begin by multiplying by the conjugate.

$$\begin{aligned}\lim_{x\to -\infty} \left(\sqrt{x^2+3x} - \sqrt{x^2-3x}\right) &= \lim_{x\to -\infty} \left(\sqrt{x^2+3x} - \sqrt{x^2-3x}\right) \frac{\sqrt{x^2+3x} + \sqrt{x^2-3x}}{\sqrt{x^2+3x} + \sqrt{x^2-3x}} \\ &= \lim_{x\to -\infty} \frac{(x^2+3x)-(x^2-3x)}{\sqrt{x^2+3x} + \sqrt{x^2-3x}} \\ &= \lim_{x\to -\infty} \frac{6x}{\sqrt{x^2+3x} + \sqrt{x^2-3x}}\end{aligned}$$

While this limit is no longer in the indeterminate form of $\infty - \infty$, it is still in an indeterminate form since both the numerator and denominator are approaching ∞. To deal with this, we now scale out a x from both the top and bottom so that those expressions no longer grow unbounded.

$$\lim_{x\to -\infty} \frac{6x}{\sqrt{x^2+3x} + \sqrt{x^2-3x}} = \lim_{x\to -\infty} \frac{6x}{\sqrt{x^2+3x} + \sqrt{x^2-3x}} \frac{1/x}{1/x}$$

$$= \lim_{x\to-\infty} \frac{6}{-\sqrt{1+3/x}-\sqrt{1-3/x}}$$
$$= \frac{6}{-1-1} = -3$$

3. (26 pts) Consider the function $f(x) = \dfrac{3x^2+6x-9}{x^2-3x+2}$.

   (a) What is the domain of f?

   (b) Find the equation of each vertical asymptote of the function $y = f(x)$, if any exist. Support your answer by evaluating the appropriate limits.

   (c) Determine the equation of each horizontal asymptote of the function $y = f(x)$, if any exist. Support your answer by evaluating the appropriate limits.

   (d) Use the definition of continuity and your work in parts (a)-(c) to identify the values of x at which $f(x)$ is discontinuous. Describe the type of discontinuity at each value.

Solution: $f(x)$ can be expressed in the following factored form:

$$f(x) = \frac{3x^2+6x-9}{x^2-3x+2} = \frac{3(x-1)(x+3)}{(x-1)(x-2)}$$

(a) The domain of a rational function consists of the set of all x values for which the polynomial in the denominator does not equal zero. In this problem, the only values of x for which the denominator equals zero are $x = 1$ and $x = 2$. Therefore, the domain of f is:

$$\boxed{x \neq 1, x \neq 2}\ , \quad \text{or equivalently,} \quad \boxed{(-\infty,1)\cup(1,2)\cup(2,\infty)}$$

(b) From part (a), there are two values of x that are not in the domain of f: $x = 1$ and $x = 2$. We evaluate the limit of $f(x)$ as x approaches each of those two values.

$$\lim_{x\to1} \frac{3(x-1)(x+3)}{(x-1)(x-2)} = \lim_{x\to1} \frac{3(x+3)}{(x-2)} = \frac{(3)(1+3)}{(1-2)} = -12$$

Since the limit of f as x approaches 1 is a finite value, there is no vertical asymptote at $x = 1$.

$$\lim_{x\to2} \frac{3(x-1)(x+3)}{(x-1)(x-2)} = \lim_{x\to2} \frac{3(x+3)}{(x-2)}$$

As x approaches 2 in the preceding expression, the numerator approaches the finite value of $(3)(2+3) = 15$ and the denominator approaches zero. Considering $x \to 2^+$, we see that as the denominator approaches zero, the denominator is positive. That is, we see that

$$\lim_{x\to2^+} \frac{3(x-1)(x+3)}{(x-1)(x-2)} = \lim_{x\to2^+} \frac{3(x+3)}{(x-2)} = \infty.$$

We could alternatively note that as $x \to 2^-$, the denominator is negative as it approaches 0. Thus,

$$\lim_{x\to2^-} \frac{3(x-1)(x+3)}{(x-1)(x-2)} = \lim_{x\to2^-} \frac{3(x+3)}{(x-2)} = -\infty.$$

Either one of these limits by themself is enough to justify that $y = f(x)$ has a vertical asymptote at $\boxed{x = 2}$.

(c) We begin by evaluating the limit of $f(x)$ as x approaches $-\infty$ and the limit of $f(x)$ as x approaches ∞.

$$\lim_{x\to-\infty} \frac{3(x-1)(x+3)}{(x-1)(x-2)} = \lim_{x\to-\infty} \frac{3(x+3)}{x-2} = \lim_{x\to-\infty} \frac{3x+9}{x-2} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to-\infty} \frac{3+\frac{9}{x}}{1-\frac{2}{x}} = \frac{3+0}{1-0} = 3$$

$$\lim_{x\to\infty} \frac{3(x-1)(x+3)}{(x-1)(x-2)} = \lim_{x\to\infty} \frac{3(x+3)}{x-2} = \lim_{x\to\infty} \frac{3x+9}{x-2} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{3+\frac{9}{x}}{1-\frac{2}{x}} = \frac{3+0}{1-0} = 3$$

Therefore, f has one horizontal asymptote: $\boxed{y = 3}$

(d) The definition of continuity indicates that a function $f(x)$ is discontinuous at $x = a$ if, and only if,

$$\lim_{x\to a} f(x) \neq f(a).$$

The fact that $f(1)$ and $f(2)$ are undefined suffices to establish that $f(x)$ is discontinuous at $\boxed{x = 1}$ and $\boxed{x = 2}$. Since a rational function is continuous wherever it is defined, those are the only two values of x at which $f(x)$ is discontinuous.

The limit of $f(x)$ as x approaches 1 and the limit of $f(x)$ as x approaches 2 were evaluated in part (b).

Since $\lim_{x\to1} f(x)$ exists, the discontinuity at $x = 1$ is $\boxed{\text{removable}}$.

Since $\lim_{x\to2^-} f(x)$ and $\lim_{x\to2^+} f(x)$ are both infinite, the discontinuity at $x = 2$ is $\boxed{\text{infinite}}$.

4. (24 pts) The following problems are not related.

(a) Suppose f and g are both odd functions and h is an even function. Furthermore, suppose f, g, and h are all defined for all real numbers. Let $j(x) = h(f(x) + g(x))$. Determine if $j(x)$ is even, odd, or neither.

(b) Consider $s(x) = \frac{x}{1-x}$ and $r(x) = \frac{x}{x+1}$. Determine $s \circ r$ and simplify as much as possible. Also, determine the domain of of $s \circ r$.

(c) Use a theorem to show that $\cos x = \frac{1}{x}$ has a solution on one of the following intervals:

- $[\frac{\pi}{2}, \pi]$
- $[-\frac{\pi}{2}, \frac{\pi}{2}]$
- $[-\pi, -\frac{\pi}{2}]$

(Be sure to state the name of the theorem that is used and justify its use. Note that you only need to show a solution exists on one of the intervals, and that some of them may not work. Be sure to identify the correct interval.)

Solution:

(a) If $j(-x) = j(x)$ then $j(x)$ is even. If $j(-x) = -j(x)$, then $j(x)$ is odd. If neither of those facts is true, then $j(x)$ is neither even nor odd. So, we test the definitions:

$$\begin{aligned} j(-x) &= h(f(-x) + g(-x)) \\ &= h(-f(x) - g(x)) \qquad \text{(because } f \text{ and } g \text{ are odd)} \\ &= h(-(f(x) + g(x))) \\ &= h(f(x) + g(x)) \qquad \text{(because } h \text{ is even)} \\ &= j(x) \end{aligned}$$

So, $j(x)$ is an even function.

(b)

$$\begin{aligned}(s\circ r)(x) &= s(r(x))\\ &= s\left(\frac{x}{x+1}\right)\\ &= \frac{\frac{x}{x+1}}{1-\frac{x}{x+1}}\\ &= \frac{\frac{x}{x+1}}{1-\frac{x}{x+1}}\cdot\frac{x+1}{x+1}\\ &= \frac{x}{x+1-x}\\ &= x,\end{aligned}$$

and the domain of $(s\circ r)(x)$ is $(-\infty,-1)\cup(-1,\infty)$.

(c) Let $f(x)=\cos x-\frac{1}{x}$. Note that $f(x)$ is continuous whenever $x\neq 0$. Also, note that $f(-\pi)=-1+\frac{1}{\pi}<0$ and $f\left(-\frac{\pi}{2}\right)=\frac{2}{\pi}>0$. So, $f(x)$ is continuous on $[-\pi,-\frac{\pi}{2}]$ and 0 is between $f(-\pi)$ and $f\left(-\frac{\pi}{2}\right)$. By the Intermediate Value Theorem, we know that $f(c)=0$ for some c in $[-\pi,-\frac{\pi}{2}]$. Thus, $\cos(c)=\frac{1}{c}$ for this c in $[-\pi,-\frac{\pi}{2}]$.

(Note that $f(x)$ is not continuous on $[-\frac{\pi}{2},\frac{\pi}{2}]$, so the intermediate value theorem does not apply to this interval. Also, $f\left(\frac{\pi}{2}\right)$ and $f(\pi)$ are both negative, so the intermediate value theorem does not imply that $f(x)=0$ along $[\frac{\pi}{2},\pi]$.)