1. (24 pts) Trigonometry

(a) A plane takes off in a straight line with an angle of inclination of $\pi/3$. How far (in the horizontal direction) will the plane have traveled when it reaches a height of 12 km?

(b) Let $f(\theta) = 6 \cos(2\theta) - 1$

i. Sketch a graph of $f$ on the interval $[0, 2\pi]$. (On your graph, label the coordinates of the $y$ intercept and the coordinates of the maximum and minimum values of the function.)

ii. What is the range of $f$?

iii. Solve the inequality $6 \cos(2\theta) - 1 > 2$ for $\theta$ in the interval $[0, 2\pi]$. Write your answer in interval notation.

Solution:

(a)

$$\tan\left(\frac{\pi}{3}\right) = \frac{\text{Height of the Plane}}{\text{Horizontal Distance Traveled}}$$

$$\sqrt{3} = \frac{12 \text{ km}}{\text{Horizontal Distance Traveled}}$$

$$4\sqrt{3} \text{ km} = \text{Horizontal Distance Traveled}$$

(b) $f(\theta) = 6 \cos(2\theta) - 1$ has period $\pi$

![Graph of f(\theta)]

i. The range of $f$ is $[-7, 5]$.

ii. Solving $6 \cos(2\theta) - 1 > 2$ is equivalent to solving $\cos(2\theta) > 1/2$. We note that for $x$ in the interval $[0, 2\pi]$, $\cos x = 1/2$ when $x = \pi/3$ or $x = 5\pi/3$. Therefore, $\cos(2\theta) = 1/2$ when $2\theta = \pi/3$ or $2\theta = 5\pi/3$. Hence, $\theta = \pi/6$ or $\theta = 5\pi/6$. Since we want to solve the inequality on the interval $[0, 2\pi]$, we use periodicity to obtain the solution $[0, \pi/6) \cup (5\pi/6, 7\pi/6) \cup (11\pi/6, 2\pi]$

2. (26 pts) Limits Evaluate the following limits and simplify your answers.

(Reminder: You may not use L’Hopital’s Rule in your solution.)

(a) $\lim_{x \to 3} \frac{|x - 3|}{2x^2 - 5x - 3}$

(b) $\lim_{x \to 0} \frac{\tan(3x) \cos(4x)}{x}$
Solution:

(a) To deal with the absolute value, we break this problem into a left limit and a right limit.

\[
\lim_{x \to -\infty} \frac{|x - 3|}{2x^2 - 5x - 3} = \lim_{x \to 3^-} \frac{-(x - 3)}{2x^2 - 5x - 3} = \lim_{x \to 3^-} \frac{-(x - 3)}{(2x + 1)(x - 3)} = \lim_{x \to 3^-} \frac{-1}{2x + 1} = -\frac{1}{7}
\]

\[
\lim_{x \to 3^+} \frac{|x - 3|}{2x^2 - 5x - 3} = \lim_{x \to 3^+} \frac{(x - 3)}{2x^2 - 5x - 3} = \lim_{x \to 3^+} \frac{(x - 3)}{(2x + 1)(x - 3)} = \lim_{x \to 3^+} \frac{1}{2x + 1} = \frac{1}{7}
\]

Since the left limit is not the same as the right limit, we conclude that the limit does not exist.

(b) When approaching a limit that involves trig functions, we aim to isolate one of the special trig limits that we already understand. In this case, \(\lim_{x \to 0} \frac{\sin(x)}{x} = 1\).

\[
\lim_{x \to 0} \frac{\tan(3x) \cos(4x)}{x} = \left(\lim_{x \to 0} \frac{\sin(3x)}{x}\right) \left(\lim_{x \to 0} \frac{\cos(4x)}{3x}\right) = 3 \left(\lim_{x \to 0} \frac{\sin(3x)}{3x}\right) (1) = 3 (1) (1) = 3
\]

(c) For a limit that involves the difference of square roots (or indeed any radicals), we begin by multiplying by the conjugate.

\[
\lim_{x \to -\infty} \left(\sqrt{x^2 + 3x} - \sqrt{x^2 - 3x}\right) = \lim_{x \to -\infty} \frac{(x^2 + 3x) - (x^2 - 3x)}{\sqrt{x^2 + 3x} + \sqrt{x^2 - 3x}} = \lim_{x \to -\infty} \frac{6x}{2x^2 + 3x + \sqrt{x^2 - 3x}} = \lim_{x \to -\infty} \frac{6x}{2x^2 + 3x + \sqrt{x^2 - 3x}} \frac{1}{x}
\]

While this limit is no longer in the indeterminate form of \(\infty - \infty\), it is still in an indeterminate form since both the numerator and denominator are approaching \(\infty\). To deal with this, we now scale out a \(x\) from both the top and bottom so that those expressions no longer grow unbounded.

\[
\lim_{x \to -\infty} \frac{6x}{\sqrt{x^2 + 3x} + \sqrt{x^2 - 3x}} = \lim_{x \to -\infty} \frac{6x}{\sqrt{x^2 + 3x} + \sqrt{x^2 - 3x}} \frac{1}{x}
\]
\[
\lim_{x \to -\infty} \frac{6}{-\sqrt{1 + 3/x} - \sqrt{1 - 3/x}} = \frac{6}{-1 - 1} = -3
\]

3. (26 pts) Consider the function \( f(x) = \frac{3x^2 + 6x - 9}{x^2 - 3x + 2} \).

(a) What is the domain of \( f \)?

(b) Find the equation of each vertical asymptote of the function \( y = f(x) \), if any exist. Support your answer by evaluating the appropriate limits.

(c) Determine the equation of each horizontal asymptote of the function \( y = f(x) \), if any exist. Support your answer by evaluating the appropriate limits.

(d) Use the definition of continuity and your work in parts (a)-(c) to identify the values of \( x \) at which \( f(x) \) is discontinuous. Describe the type of discontinuity at each value.

Solution: \( f(x) \) can be expressed in the following factored form:

\[
f(x) = \frac{3x^2 + 6x - 9}{x^2 - 3x + 2} = \frac{3(x - 1)(x + 3)}{(x - 1)(x - 2)}
\]

(a) The domain of a rational function consists of the set of all \( x \) values for which the polynomial in the denominator does not equal zero. In this problem, the only values of \( x \) for which the denominator equals zero are \( x = 1 \) and \( x = 2 \). Therefore, the domain of \( f \) is:

\[
x \neq 1, x \neq 2 \quad \text{or equivalently, } (-\infty, 1) \cup (1, 2) \cup (2, \infty)
\]

(b) From part (a), there are two values of \( x \) that are not in the domain of \( f \): \( x = 1 \) and \( x = 2 \). We evaluate the limit of \( f(x) \) as \( x \) approaches each of those two values.

\[
\lim_{x \to 1} \frac{3(x - 1)(x + 3)}{(x - 1)(x - 2)} = \lim_{x \to 1} \frac{3(x + 3)}{(x - 2)} = \frac{(3)(1 + 3)}{(1 - 2)} = -12
\]

Since the limit of \( f \) as \( x \) approaches 1 is a finite value, there is no vertical asymptote at \( x = 1 \).

\[
\lim_{x \to 2} \frac{3(x - 1)(x + 3)}{(x - 1)(x - 2)} = \lim_{x \to 2} \frac{3(x + 3)}{(x - 2)}
\]

As \( x \) approaches 2 in the preceding expression, the numerator approaches the finite value of \( (3)(2 + 3) = 15 \) and the denominator approaches zero. Considering \( x \to 2^+ \), we see that as the denominator approaches zero, the denominator is positive. That is, we see that

\[
\lim_{x \to 2^+} \frac{3(x - 1)(x + 3)}{(x - 1)(x - 2)} = \lim_{x \to 2^+} \frac{3(x + 3)}{(x - 2)} = \infty.
\]

We could alternatively note that as \( x \to 2^- \), the denominator is negative as it approaches 0. Thus,

\[
\lim_{x \to 2^-} \frac{3(x - 1)(x + 3)}{(x - 1)(x - 2)} = \lim_{x \to 2^-} \frac{3(x + 3)}{(x - 2)} = -\infty.
\]
Either one of these limits by themself is enough to justify that $y = f(x)$ has a vertical asymptote at $x = 2$.

(c) We begin by evaluating the limit of $f(x)$ as $x$ approaches $-\infty$ and the limit of $f(x)$ as $x$ approaches $\infty$.

$$
\lim_{x \to -\infty} \frac{3(x-1)(x+3)}{(x-1)(x-2)} = \lim_{x \to -\infty} \frac{3(x+3)}{x-2} = \lim_{x \to -\infty} \frac{3x + 9}{x - 2} \cdot \frac{1}{x} = \lim_{x \to -\infty} \frac{3 + \frac{9}{x}}{1 - \frac{2}{x}} = \frac{3 + 0}{1 - 0} = 3
$$

$$
\lim_{x \to \infty} \frac{3(x-1)(x+3)}{(x-1)(x-2)} = \lim_{x \to \infty} \frac{3(x+3)}{x-2} = \lim_{x \to \infty} \frac{3x + 9}{x - 2} \cdot \frac{1}{x} = \lim_{x \to \infty} \frac{3 + \frac{9}{x}}{1 - \frac{2}{x}} = \frac{3 + 0}{1 - 0} = 3
$$

Therefore, $f$ has one horizontal asymptote: $y = 3$.

(d) The definition of continuity indicates that a function $f(x)$ is discontinuous at $x = a$ if, and only if,

$$
\lim_{x \to a} f(x) \neq f(a).
$$

The fact that $f(1)$ and $f(2)$ are undefined suffices to establish that $f(x)$ is discontinuous at $x = 1$ and $x = 2$. Since a rational function is continuous wherever it is defined, those are the only two values of $x$ at which $f(x)$ is discontinuous.

The limit of $f(x)$ as $x$ approaches 1 and the limit of $f(x)$ as $x$ approaches 2 were evaluated in part (b).

Since $\lim_{x \to 1} f(x)$ exists, the discontinuity at $x = 1$ is removable.

Since $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$ are both infinite, the discontinuity at $x = 2$ is infinite.

4. (24 pts) The following problems are not related.

(a) Suppose $f$ and $g$ are both odd functions and $h$ is an even function. Furthermore, suppose $f$, $g$, and $h$ are all defined for all real numbers. Let $j(x) = h(f(x) + g(x))$. Determine if $j(x)$ is even, odd, or neither.

(b) Consider $s(x) = \frac{x}{1-x}$ and $r(x) = \frac{x}{x+1}$. Determine $s \circ r$ and simplify as much as possible. Also, determine the domain of $s \circ r$.

(c) Use a theorem to show that $\cos x = \frac{1}{2}$ has a solution on one of the following intervals:
- $\left[\frac{\pi}{2}, \pi\right]$  
- $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$  
- $[-\pi, -\frac{\pi}{2}]$

(Be sure to state the name of the theorem that is used and justify its use. Note that you only need to show a solution exists on one of the intervals, and that some of them may not work. Be sure to identify the correct interval.)

**Solution:**

(a) If $j(-x) = j(x)$ then $j(x)$ is even. If $j(-x) = -j(x)$, then $j(x)$ is odd. If neither of those facts is true, then $j(x)$ is neither even nor odd. So, we test the definitions:

$$
\begin{align*}
    j(-x) &= h(f(-x) + g(-x)) \\
         &= h(-f(x) - g(x)) \quad \text{(because $f$ and $g$ are odd)} \\
         &= h(-(f(x) + g(x))) \\
         &= h(f(x) + g(x)) \quad \text{(because $h$ is even)} \\
         &= j(x)
\end{align*}
$$

So, $j(x)$ is an even function.
(b) 

\[(s \circ r)(x) = s(r(x))\]

\[= s\left(\frac{x}{x+1}\right)\]

\[= \frac{x}{x+1} \cdot \frac{x + 1}{x + 1}\]

\[= \frac{x}{x + 1 - x}\]

\[= x,\]

and the domain of \((s \circ r)(x)\) is \((-\infty, -1) \cup (-1, \infty)\).

(c) Let \(f(x) = \cos x - \frac{1}{x}\). Note that \(f(x)\) is continuous whenever \(x \neq 0\). Also, note that \(f(-\pi) = -1 + \frac{1}{\pi} < 0\) and \(f\left(-\frac{\pi}{2}\right) = \frac{2}{\pi} > 0\). So, \(f(x)\) is continuous on \([-\pi, -\frac{\pi}{2}]\) and 0 is between \(f(-\pi)\) and \(f\left(-\frac{\pi}{2}\right)\). By the Intermediate Value Theorem, we know that \(f(c) = 0\) for some \(c\) in \([-\pi, -\frac{\pi}{2}]\). Thus, \(\cos(c) = \frac{1}{c}\) for this \(c\) in \([-\pi, -\frac{\pi}{2}]\).

(Note that \(f(x)\) is not continuous on \([-\frac{\pi}{2}, \frac{\pi}{2}]\), so the intermediate value theorem does not apply to this interval. Also, \(f\left(\frac{\pi}{2}\right)\) and \(f(\pi)\) are both negative, so the intermediate value theorem does not imply that \(f(x) = 0\) along \([\frac{\pi}{2}, \pi]\).)