1. (38 points) Find the requested information. Justify your answer.

   (a) Simplify $\cos^{-1}(\cos(5\pi/4))$

   (b) Simplify $\tan(\sin^{-1}(2x))$

   (c) Write the following sum in sigma notation: $2 - \frac{3}{8} + \frac{4}{27} - \frac{5}{64} + \frac{6}{125} - \frac{7}{216}$.

   (d) $\lim_{x \to 0} \frac{x7^x}{7^x - 1}$

   (e) Find dy/dx for $y = (\ln x)^{\sin 3x}$ (Find dy/dx in terms of x, but you need not simplify your answer further.)

Solution:

(a) Note that $\frac{5\pi}{4}$ does not lie in $[0, \pi]$, the restriction to the domain of cosine that is used to define inverse-cosine. So, we use the unit circle to see the following:

$$\cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right) = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \boxed{\frac{3\pi}{4}}$$

(b) Let $\theta = \sin^{-1}(2x)$. Then, we have $\sin\theta = 2x$ where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. So, we need only find $\tan\theta$. If we consider a right-triangle with angle θ, then we can assume the opposite side is $2x$ and the hypotenuse is 1. Using the Pythagorean theorem tells us that the adjacent side must be $\sqrt{1 - 4x^2}$. (We know this quantity is positive because θ lies in quadrant I or IV.) So,

$$\tan(\sin^{-1}(2x)) = \tan\theta = \boxed{\frac{2x}{\sqrt{1 - 4x^2}}}$$

(c) $$2 - \frac{3}{8} + \frac{4}{27} - \frac{5}{64} + \frac{6}{125} - \frac{7}{216} = \boxed{\sum_{k=1}^{6} \frac{(-1)^{k+1}(k+1)}{k^3}}$$

(d) Note that $\lim_{x \to 0} \frac{x7^x}{7^x - 1}$ is a $\frac{0}{0}$ indeterminate form. So, L'Hospital's Rule applies:

$$\begin{aligned}
\lim_{x \to 0} \frac{x7^x}{7^x - 1} &= \lim_{x \to 0} \frac{x7^x \ln 7 + 7^x}{7^x \ln 7} \\
&= \lim_{x \to 0} \frac{x \ln 7 + 1}{\ln 7} \\
&= \boxed{\frac{1}{\ln 7}}
\end{aligned}$$

(e) There are two ways to find the derivative.

i. Solution 1: Rewrite y using the fact that the exponential and $\ln$ are inverses of each other:

$$y = (\ln x)^{\sin(3x)} = e^{\ln((\ln x)^{\sin(3x)})} = e^{(\sin(3x))\ln(\ln x)}$$

Then, we can differentiate:

$$\frac{dy}{dx} = e^{(\sin(3x))\ln(\ln x)}\left[\sin(3x)\frac{1}{x\ln x} + 3\cos(3x)\ln(\ln x)\right]$$

Rewriting:

$$\frac{dy}{dx} = (\ln x)^{\sin 3x}\left(\frac{\sin(3x)}{x\ln x} + 3\cos(3x)\ln(\ln x)\right)$$

ii. Solution 2: Use logarithmic differentiation. That is, apply the natural logarithm to both sides, use the power property of logarithms to bring the power down, and then differentiate both sides.

$$\begin{aligned}
y &= (\ln x)^{\sin 3x} \\
\ln y &= \ln(\ln x)^{\sin 3x} \\
\ln y &= \sin(3x)\ln(\ln x) \\
\frac{d}{dx}(\ln y) &= \frac{d}{dx}(\sin(3x)\ln(\ln x)) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{\sin(3x)}{x\ln x} + 3\cos(3x)\ln(\ln x) \\
\frac{dy}{dx} &= (\ln x)^{\sin 3x}\left(\frac{\sin(3x)}{x\ln x} + 3\cos(3x)\ln(\ln x)\right)
\end{aligned}$$

2. (20 points) Evaluate the following integrals. Simplify your answer.

(a) $\displaystyle\int_0^{\ln 2} \tanh x\, dx$

(b) $\displaystyle\int_1^{3^{1/4}} \frac{t}{1+t^4}\, dt$ (Note: If you find the upper limit of this integral hard to read, it is 3 raised to the $1/4$ power.)

Solution:

(a) We know

$$\int_0^{\ln 2} \tanh x\, dx = \int_0^{\ln 2} \frac{\sinh x}{\cosh x}\, dx.$$

So, we will use the substitution $u = \cosh x$ with $du = \sinh x dx$. When $x = 0$ then $u = \cosh(0) = 1$ and when $x = \ln 2$ then $u = \cosh(\ln 2) = \dfrac{e^{\ln 2} + e^{-\ln 2}}{2} = \dfrac{5}{4}$

So,

$$\int_0^{\ln 2} \tanh x\, dx = \int_1^{5/4} \frac{1}{u}\, du$$
$$= \ln\left(\frac{5}{4}\right) - \ln(1)$$
$$= \boxed{\ln\left(\frac{5}{4}\right)}$$

(b) We will use the substitution $u = t^2$. So, $du = 2tdt$. It will be more useful to write $\frac{1}{2}du = tdt$. Also, we see that when $t = 1$ then $u = 1^2 = 1$ and when $t = 3^{\frac{1}{4}}$ then $u = (3^{\frac{1}{4}})^2 = \sqrt{3}$.

So,

$$\int_1^{3^{1/4}} \frac{t}{1+t^4}\, dt = \frac{1}{2}\int_1^{\sqrt{3}} \frac{1}{1+u^2} du$$
$$= \frac{1}{2}\Big(\arctan(\sqrt{3}) - \arctan(1)\Big)$$
$$= \frac{1}{2}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$
$$= \boxed{\frac{\pi}{24}}$$

3. (30 points) Three unrelated problems. Explain your work and state any theorem that you use.

(a) Suppose that $2 \leq g'(x) \leq 5$ is true for all x and that $g(0) = a$. Find a reasonable upper and lower bound for $g(4)$. Briefly explain your reasoning.

(b) Find an upper and lower bound for $\int_{-3}^{-1} \frac{t^2}{1+t^2}\, dt$ by finding the upper Riemann sum and the lower Riemann sum with $n = 1$.

(c) Let $f(x) = \int_1^x \frac{t^2}{t^2+t+2}\, dt$ and answer the following two questions.

i. Find the linearization of f at $x = 1$ and use this to approximate $\int_1^{1.1} \frac{t^2}{t^2+t+2}\, dt$

ii. Find the interval(s) where $f(x)$ is concave down.

Solution:

(a) We are given that $g(x)$ is differentiable everywhere. Thus, $g(x)$ is continuous everywhere. So, the Mean Value Theorem for Derivative applies to this function on any closed interval. So, there exists c in (0,4) such that

$$g'(c) = \frac{g(4) - g(0)}{4 - 0} = \frac{g(4) - a}{4}.$$

This means that

$$2 \leq \frac{g(4) - a}{4} \leq 5.$$

If we solve for $g(4)$, we see that

$$8 \leq g(4) - a \leq 20$$

which gives

$$\boxed{8 + a \leq g(4) \leq 20 + a}$$

Note: Other justifications were also accepted.

(b) Note that

$$\begin{aligned} \frac{d}{dt}\left(\frac{t^2}{1+t^2}\right) &= \frac{(1+t^2)2t - t^2(2t)}{(1+t^2)^2} \\ &= \frac{2t}{(1+t^2)^2} \end{aligned}$$

is always negative on $[-3, -1]$. So, the integrand is a decreasing function. Thus, we know that the absolute minimum of the integrand on $[-3, -1]$ is $\frac{(-1)^2}{1+(-1)^2} = \frac{1}{2}$ and the absolute maximum of the integrand on $[-3, -1]$ is $\frac{(-3)^2}{1+(-3)^2} = \frac{9}{10}$. So, we have

$$\frac{1}{2}(-1 - (-3)) \leq \int_{-3}^{-1} \frac{t^2}{1+t^2}\, dt \leq \frac{9}{10}(-1 - (-3)),$$

which simplifies to

$$\boxed{1 \leq \int_{-3}^{-1} \frac{t^2}{1+t^2}\, dt \leq \frac{9}{5}}$$

(c) The problem requires the Fundamental Theorem of Calculus.

i. We first observe that $f(1) = 0$ and $f'(x) = \dfrac{x^2}{x^2+x+2}$. So, $f'(1) = 1/4$. The linearization is given by $L(x) = f(1) + f'(1)(x-1)$. So, we have $\boxed{L(x) = (1/4)(x-1)}$

Next, we note

$$f(1.1) = \int_1^{1.1} \frac{t^2}{t^2+t+2}\, dt \approx L(1.1) = (1/4)(1.1 - 1) = \boxed{\frac{1}{40}}$$

ii. To find the interval where f is concave down, we compute

$$f''(x) = \frac{(x^2+x+2)(2x) - x^2(2x+1)}{(x^2+x+2)^2} = \frac{x^2+4x}{(x^2+x+2)^2} = \frac{x(x+4)}{(x^2+x+2)^2}$$

Since the denominator is always positive, $f''(x)$ has two critical point at $x = -4$ and $x = 0$. Test any point in the interval $(-\infty, -4)$ to see that $f''(x) > 0$. For any point in the interval $(-4, 0)$ we have $f''(x) < 0$ and $f''(x) > 0$ for $x > 0$. Thus, we see that f is concave down on the interval $\boxed{(-4, 0)}$.

4. (15 points) Find the (x, y) coordinates of the local and absolute maxima and minima of $f(x) = x\sqrt{1-x}$ on the interval $[-1, 1]$. If a local or absolute max or min does not exist, state this.
Solution: Apply the product rule and then get a common denominator to obtain

$$f'(x) = (1-x)^{1/2} - (x/2)(1-x)^{-1/2} = (1-x)^{1/2} - \frac{x}{2(1-x)^{1/2}} = \frac{2-3x}{2(1-x)^{1/2}}$$

Note that $f'(x)$ exists on $(-1, 1)$. So, the only critical point is $x = 2/3$ which occurs when $f'(x) = 0$.

Since $f'(x) > 0$ if $-1 < x < \frac{2}{3}$ and $f'(x) < 0$ if $\frac{2}{3} < x < 1$, then the first derivative test implies that $f(x)$ has a local maximum at $x = \frac{2}{3}$ of $f\left(\frac{2}{3}\right) = \frac{2}{3\sqrt{3}}$. There is no local minimum.

Since f is a continuous function on the closed interval $[-1, 1]$, the Extreme Value Theorem guarantees the existence of both an absolute max and and an absolute min. We see that

$$f(-1) = -\sqrt{2} \qquad f\left(\frac{2}{3}\right) = \frac{2}{3\sqrt{3}} \qquad f(1) = 0.$$

So, the absolute maximum is $\frac{2}{3\sqrt{3}}$ at $x = \frac{2}{3}$ and the absolute minimum is $-\sqrt{2}$ at $x = -1$.

5. (12 points) You watch the vertical launch of a rocket from 5 km away. Let θ be the angle between the ground and your view of the rocket. This angle, called the angle of elevation above the horizon, increases at a rate of $\pi/30$ rad/sec when $\theta = \pi/4$. Find the speed of the rocket at this instant.

Solution:

Let h be the height of the rocket t seconds after launch. Note that $\tan\theta = \dfrac{h}{5}$. Since h and θ are functions of t, we differentiate both sides with respect to t, then we have $\sec^2\theta \dfrac{d\theta}{dt} = \dfrac{1}{5}\dfrac{dh}{dt}$. Thus, we have

$$\frac{dh}{dt} = 5\sec^2\theta \tfrac{d\theta}{dt}$$

Evaluating at $\theta = \pi/4$ we obtain

$$\frac{dh}{dt}\Big|_{\theta=\pi/4} = 5(\sqrt{2})^2 \frac{\pi}{30} = \boxed{\frac{\pi}{3} \text{ km per second.}}$$

6. (15 points) Consider $f(x) = \dfrac{e^x}{e^x + 1}$.

   (a) Show $f(x)$ is one-to-one.

   (b) Find the inverse of $f(x)$.

Solution:

(a) Note that

$$f'(x) = \frac{(e^x + 1)e^x - e^x(e^x)}{(e^x + 1)^2} = \frac{e^x}{(e^x + 1)^2}$$

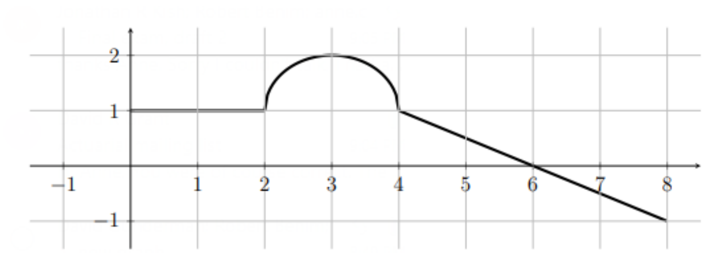
is always positive. So, $f(x)$ is always increasing and must be one-to-one.

(b) To find the inverse, we set $x = \dfrac{e^y}{e^y + 1}$ and solve for y.

$$\begin{aligned} x &= \frac{e^y}{e^y + 1} \\ x(e^y + 1) &= e^y \\ xe^y + x &= e^y \\ xe^y - e^y &= -x \\ e^y(x - 1) &= -x \\ e^y &= \frac{x}{1 - x} \\ y &= \ln\left(\frac{x}{1 - x}\right) \end{aligned}$$

Thus, $\boxed{f^{-1}(x) = \ln\left(\dfrac{x}{1 - x}\right).}$

7. (20 points) Consider the function $f(x)$ defined over $[0, 8]$ that is graphed below. It consists of two straight line segments and a semicircle.



(a) Find the average value of f over the interval $[0, 8]$.

(b) Evaluate $\lim\limits_{h \to 0} \dfrac{f(5 + h) - f(5)}{h}$.

(c) Show that $h(x) = \dfrac{f(x)}{(x - 6)^2}$ has a vertical asymptote at $x = 6$.

Solution:

(a) By inspecting the graph, we obtain:

$$f_{avg} = \frac{1}{8} \int_0^8 f(x)\, dx = \boxed{\frac{1}{8}\left(4 + \frac{\pi}{2}\right)}$$

(b) Note that the slope at $x = 5$ is $-1/2$ and $f'(5) = \lim_{h\to 0} \frac{f(5+h) - f(5)}{h}$ so

$$\boxed{\lim_{h\to 0} \frac{f(5+h) - f(5)}{h} = \frac{-1}{2}}$$

(c) To show that h has a vertical asymptote at $x = 6$, we need to calculate either $\lim_{x\to 6^+} \frac{f(x)}{(x-6)^2} = -\infty$ or $\lim_{x\to 6^-} \frac{f(x)}{(x-6)^2} = \infty$. There are two possible ways to do this. One is to use L'Hospital's rule and the other is to find that the straight line segment of f for $4 < x < 8$ is given by $y = 3 - x/2 = (6-x)/2$ and evaluate

$$\lim_{x\to 6^+} \frac{f(x)}{(x-6)^2} = \lim_{x\to 6^+} \frac{(6-x)/2}{(x-6)^2} = \lim_{x\to 6^+} \frac{-1}{2(x-6)} = -\infty$$

The limit as x approaches 6 from the left can be calculated in a similar manner.

8. At the bottom of your work for problem 7: please write a statement that says your work is your own and you did not receive any help on the exam. Sign this statement. **Verify that everything has been uploaded correctly and pages have been associated to the correct problem before you leave Proctorio or the zoom proctoring room.**