1. (38 points) Find the requested information. Justify your answer.

(a) Simplify \( \cos^{-1}(\cos(5\pi/4)) \)

(b) Simplify \( \tan(\sin^{-1}(2x)) \)

(c) Write the following sum in sigma notation: \( 2 - \frac{3}{8} + \frac{4}{27} - \frac{5}{64} + \frac{6}{125} - \frac{7}{216} \).

(d) \( \lim_{x \to 0} \frac{x^7}{7^x - 1} \)

(e) Find \( dy/dx \) for \( y = (\ln x)^{\sin 3x} \) (Find \( dy/dx \) in terms of \( x \), but you need not simplify your answer further.)

Solution:

(a) Note that \( \frac{5\pi}{4} \) does not lie in \([0, \pi]\), the restriction to the domain of cosine that is used to define inverse-cosine. So, we use the unit circle to see the following:

\[
\cos^{-1} \left( \cos \left( \frac{5\pi}{4} \right) \right) = \cos^{-1} \left( -\frac{\sqrt{2}}{2} \right) = \frac{3\pi}{4}
\]

(b) Let \( \theta = \sin^{-1}(2x) \). Then, we have \( \sin \theta = 2x \) where \( -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \). So, we need only find \( \tan \theta \). If we consider a right-triangle with angle \( \theta \), then we can assume the opposite side is \( 2x \) and the hypotenuse is 1. Using the Pythagorean theorem tells us that the adjacent side must be \( \sqrt{1 - 4x^2} \). (We know this quantity is positive because \( \theta \) lies in quadrant I or IV.) So,

\[
\tan(\sin^{-1}(2x)) = \tan \theta = \frac{2x}{\sqrt{1 - 4x^2}}
\]

(c) \( 2 - \frac{3}{8} + \frac{4}{27} - \frac{5}{64} + \frac{6}{125} - \frac{7}{216} = \sum_{k=1}^{6} \frac{(-1)^{k+1}(k + 1)}{k^3} \)

(d) Note that \( \lim_{x \to 0} \frac{x^7}{7^x - 1} \) is a \( \frac{0}{0} \) indeterminate form. So, L’Hospital’s Rule applies:

\[
\lim_{x \to 0} \frac{x^7}{7^x - 1} = \lim_{x \to 0} \frac{x^7 \ln 7 + 7^x}{7^x \ln 7}
\]

\[
= \lim_{x \to 0} \frac{x \ln 7 + 1}{\ln 7}
\]

\[
= \frac{1}{\ln 7}
\]

(e) There are two ways to find the derivative.
i. Solution 1: Rewrite $y$ using the fact that the exponential and $\ln$ are inverses of each other:

$$y = (\ln x)^{\sin(3x)} = e^{\ln((\ln x)^{\sin(3x)})} = e^{(\sin(3x)) \ln(\ln x)}$$

Then, we can differentiate:

$$\frac{dy}{dx} = e^{(\sin(3x)) \ln(\ln x)} \left[ \sin(3x) \frac{1}{x \ln x} + 3 \cos(3x) \ln(\ln x) \right]$$

Rewriting:

$$\frac{dy}{dx} = (\ln x)^{\sin(3x)} \left( \frac{\sin(3x)}{x \ln x} + 3 \cos(3x) \ln(\ln x) \right)$$

ii. Solution 2: Use logarithmic differentiation. That is, apply the natural logarithm to both sides, use the power property of logarithms to bring the power down, and then differentiate both sides.

$$y = (\ln x)^{\sin(3x)}$$

$$\ln y = \ln((\ln x)^{\sin(3x)})$$

$$\ln y = \sin(3x) \ln(\ln x)$$

$$\frac{d}{dx} (\ln y) = \frac{d}{dx} (\sin(3x) \ln(\ln x))$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{\sin(3x)}{x \ln x} + 3 \cos(3x) \ln(\ln x)$$

$$\frac{dy}{dx} = (\ln x)^{\sin(3x)} \left( \frac{\sin(3x)}{x \ln x} + 3 \cos(3x) \ln(\ln x) \right)$$

2. (20 points) Evaluate the following integrals. Simplify your answer.

(a) \[ \int_{0}^{\ln 2} \tanh x \, dx \]

(b) \[ \int_{1}^{3^{1/4}} \frac{t}{1 + t^4} \, dt \] (Note: If you find the upper limit of this integral hard to read, it is 3 raised to the 1/4 power.)

Solution:

(a) We know

$$\int_{0}^{\ln 2} \tanh x \, dx = \int_{0}^{\ln 2} \frac{\sinh x}{\cosh x} \, dx.$$

So, we will use the substitution $u = \cosh x$ with $du = \sinh x \, dx$. When $x = 0$ then $u = \cosh(0) = 1$ and when $x = \ln 2$ then $u = \cosh(\ln 2) = \frac{e^{\ln 2} + e^{-\ln 2}}{2} = \frac{5}{4}$.

So,
\[ \int_0^{\ln 2} \tanh x \, dx = \int_1^{5/4} \frac{1}{u} \, du \]

\[ = \ln \left( \frac{5}{4} \right) - \ln(1) \]

\[ = \ln \left( \frac{5}{4} \right) \]

(b) We will use the substitution \( u = t^2 \). So, \( du = 2t \, dt \). It will be more useful to write \( \frac{1}{2}du = t \, dt \). Also, we see that when \( t = 1 \) then \( u = 1^2 = 1 \) and when \( t = 3^{1/4} \) then \( u = (3^{1/4})^2 = \sqrt{3} \).

So,

\[ \int_1^{3^{1/4}} \frac{t}{1 + t^4} \, dt = \frac{1}{2} \int_1^{\sqrt{3}} \frac{1}{1 + u^2} \, du \]

\[ = \frac{1}{2} \left( \arctan(\sqrt{3}) - \arctan(1) \right) \]

\[ = \frac{1}{2} \left( \frac{\pi}{3} - \frac{\pi}{4} \right) \]

\[ = \frac{\pi}{24} \]

3. (30 points) Three unrelated problems. Explain your work and state any theorem that you use.

(a) Suppose that \( 2 \leq g'(x) \leq 5 \) is true for all \( x \) and that \( g(0) = a \). Find a reasonable upper and lower bound for \( g(4) \). Briefly explain your reasoning.

(b) Find an upper and lower bound for \( \int_{-3}^{-1} \frac{t^2}{1 + t^2} \, dt \) by finding the upper Riemann sum and the lower Riemann sum with \( n = 1 \).

(c) Let \( f(x) = \int_1^x \frac{t^2}{t^2 + t + 2} \, dt \) and answer the following two questions.

i. Find the linearization of \( f \) at \( x = 1 \) and use this to approximate \( \int_1^{1.1} \frac{t^2}{t^2 + t + 2} \, dt \)

ii. Find the interval(s) where \( f(x) \) is concave down.

Solution:

(a) We are given that \( g(x) \) is differentiable everywhere. Thus, \( g(x) \) is continuous everywhere. So, the Mean Value Theorem for Derivative applies to this function on any closed interval. So, there exists \( c \) in \( (0,4) \) such that

\[ g'(c) = \frac{g(4) - g(0)}{4 - 0} = \frac{g(4) - a}{4} \]
This means that 
\[2 \leq \frac{g(4) - a}{4} \leq 5.\]

If we solve for \(g(4)\), we see that 
\[8 \leq g(4) - a \leq 20\]
which gives 
\[8 + a \leq g(4) \leq 20 + a\]

Note: Other justifications were also accepted.

(b) Note that 
\[\frac{d}{dt} \left( \frac{t^2}{1 + t^2} \right) = \frac{(1 + t^2)2t - t^2(2t)}{(1 + t^2)^2} = \frac{2t}{(1 + t^2)^2}\]
is always negative on \([-3, -1]\). So, the integrand is a decreasing function. Thus, we know that the absolute minimum of the integrand on \([-3, -1]\) is \(\frac{(-1)^2}{1+(-1)^2} = \frac{1}{2}\) and the absolute maximum of the integrand on \([-3, -1]\) is \(\frac{(-3)^2}{1+(-3)^2} = \frac{9}{10}\). So, we have 
\[\frac{1}{2}(-1 - (-3)) \leq \int_{-3}^{-1} \frac{t^2}{1 + t^2} \, dt \leq \frac{9}{10}(-1 - (-3)),\]
which simplifies to
\[1 \leq \int_{-3}^{-1} \frac{t^2}{1 + t^2} \, dt \leq \frac{9}{5}\]

(c) The problem requires the Fundamental Theorem of Calculus.

i. We first observe that \(f(1) = 0\) and \(f'(x) = \frac{x^2}{x^2 + x + 2}\). So, \(f'(1) = 1/4\). The linearization is given by \(L(x) = f(1) + f'(1)(x-1)\). So, we have \(L(x) = (1/4)(x-1)\)

Next, we note 
\[f(1.1) = \int_{1}^{1.1} \frac{t^2}{t^2 + t + 2} \, dt \approx L(1.1) = (1/4)(1.1 - 1) = \frac{1}{40}\]

ii. To find the interval where \(f\) is concave down, we compute 
\[f''(x) = \frac{(x^2 + x + 2)(2x) - x^2(2x + 1)}{(x^2 + x + 2)^2} = \frac{x^2 + 4x}{(x^2 + x + 2)^2} = \frac{x(x+4)}{(x^2 + x + 2)^2}\]

Since the denominator is always positive, \(f''(x)\) has two critical point at \(x = -4\) and \(x = 0\). Test any point in the interval \((-\infty, -4)\) to see that \(f''(x) > 0\). For any point in the interval \((-4, 0)\) we have \(f''(x) < 0\) and \(f''(x) > 0\) for \(x > 0\). Thus, we see that \(f\) is concave down on the interval \([-4, 0]\).
4. (15 points) Find the \((x,y)\) coordinates of the local and absolute maxima and minima of \(f(x) = x\sqrt{1-x}\) on the interval \([-1,1]\). If a local or absolute max or min does not exist, state this.

**Solution:** Apply the product rule and then get a common denominator to obtain

\[
f'(x) = (1-x)^{1/2} - (x/2)(1-x)^{-1/2} = (1-x)^{1/2} - \frac{x}{2(1-x)^{1/2}} = \frac{2-3x}{2(1-x)^{1/2}}
\]

Note that \(f'(x)\) exists on \((-1,1)\). So, the only critical point is \(x = 2/3\) which occurs when \(f'(x) = 0\).

Since \(f'(x) > 0\) if \(-1 < x < \frac{2}{3}\) and \(f'(x) < 0\) if \(\frac{2}{3} < x < 1\), then the first derivative test implies that \(f(x)\) has a local maximum at \(x = \frac{2}{3}\) of \(f\left(\frac{2}{3}\right) = \frac{2}{3\sqrt{3}}\). There is no local minimum.

Since \(f\) is a continuous function on the closed interval \([-1,1]\), the Extreme Value Theorem guarantees the existence of both an absolute max and and an absolute min. We see that

\[
f(-1) = -\sqrt{2} \quad f\left(\frac{2}{3}\right) = \frac{2}{3\sqrt{3}} \quad f(1) = 0.
\]

So, the absolute maximum is \(\frac{2}{3\sqrt{3}}\) at \(x = \frac{2}{3}\) and the absolute minimum is \(-\sqrt{2}\) at \(x = -1\).

5. (12 points) You watch the vertical launch of a rocket from 5 km away. Let \(\theta\) be the angle between the ground and your view of the rocket. This angle, called the angle of elevation above the horizon, increases at a rate of \(\pi/30\) rad/sec when \(\theta = \pi/4\). Find the speed of the rocket at this instant.

**Solution:**

Let \(h\) be the height of the rocket \(t\) seconds after launch. Note that \(\tan \theta = \frac{h}{5}\). Since \(h\) and \(\theta\) are functions of \(t\), we differentiate both sides with respect to \(t\), then we have \(\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{5} \frac{dh}{dt}\).

Thus, we have

\[
\frac{dh}{dt} = 5 \sec^2 \theta \frac{d\theta}{dt}
\]

Evaluating at \(\theta = \pi/4\) we obtain

\[
\frac{dh}{dt} \bigg|_{\theta=\pi/4} = 5(\sqrt{2})^2 \frac{\pi}{30} = \frac{\pi}{3} \text{ km per second.}
\]

6. (15 points) Consider \(f(x) = \frac{e^x}{e^x + 1}\).

(a) Show \(f(x)\) is one-to-one.

(b) Find the inverse of \(f(x)\).

**Solution:**
(a) Note that
\[
f'(x) = \frac{(e^x + 1)e^x - e^x(e^x)}{(e^x + 1)^2} = \frac{e^x}{(e^x + 1)^2}
\]
is always positive. So, \( f(x) \) is always increasing and must be one-to-one.

(b) To find the inverse, we set \( x = \frac{e^y}{e^y + 1} \) and solve for \( y \).

\[
\begin{align*}
x &= \frac{e^y}{e^y + 1} \\
x(e^y + 1) &= e^y \\
x e^y + x &= e^y \\
x e^y - e^y &= -x \\
e^y(x - 1) &= -x \\
e^y &= \frac{x}{1 - x} \\
y &= \ln \left( \frac{x}{1 - x} \right)
\end{align*}
\]

Thus, \( f^{-1}(x) = \ln \left( \frac{x}{1 - x} \right) \).

7. (20 points) Consider the function \( f(x) \) defined over \([0, 8]\) that is graphed below. It consists of two straight line segments and a semicircle.

(a) Find the average value of \( f \) over the interval \([0, 8]\).

(b) Evaluate \( \lim_{h \to 0} \frac{f(5 + h) - f(5)}{h} \).

(c) Show that \( h(x) = \frac{f(x)}{(x - 6)^2} \) has a vertical asymptote at \( x = 6 \).

Solution:

(a) By inspecting the graph, we obtain:

\[
\begin{align*}
f_{avg} &= \frac{1}{8} \int_0^8 f(x) \, dx = \frac{1}{8} \left( 4 + \frac{\pi}{2} \right)
\end{align*}
\]
(b) Note that the slope at \( x = 5 \) is \(-1/2\) and \( f'(5) = \lim_{h \to 0} \frac{f(5 + h) - f(5)}{h} \) so

\[
\lim_{h \to 0} \frac{f(5 + h) - f(5)}{h} = \frac{-1}{2}
\]

(c) To show that \( h \) has a vertical asymptote at \( x = 6\), we need to calculate either \( \lim_{x \to 6^+} \frac{f(x)}{(x - 6)^2} = -\infty \) or \( \lim_{x \to 6^-} \frac{f(x)}{(x - 6)^2} = \infty \). There are two possible ways to do this. One is to use L’Hospital’s rule and the other is to find that the straight line segment of \( f \) for \( 4 < x < 8 \) is given by \( y = 3 - x/2 = (6 - x)/2 \) and evaluate

\[
\lim_{x \to 6^+} \frac{f(x)}{(x - 6)^2} = \lim_{x \to 6^+} \frac{(6 - x)/2}{(x - 6)^2} = \lim_{x \to 6^+} \frac{-1}{2(x - 6)} = -\infty
\]

The limit as \( x \) approaches 6 from the left can be calculated in a similar manner.

8. At the bottom of your work for problem 7: please write a statement that says your work is your own and you did not receive any help on the exam. Sign this statement. Verify that everything has been uploaded correctly and pages have been associated to the correct problem before you leave Proctorio or the zoom proctoring room.