1. (22 points) Find the requested information, if possible. If it is not possible, explain.

   (a) Find $s(t)$ if $s'(t) = \sqrt{t} - 3\sin t$ and $s(0) = 4$.

   (b) $\displaystyle\int_{-2}^{1} \frac{1}{x^2} dx$

   (c) $\displaystyle\int_0^2 \frac{ax^2+3b}{(ax^3+9bx+1)^{1/3}}\, dx$, a and b are positive constants.

   Solution:

   (a)

   $$s(t) = (2/3)t^{3/2} + 3\cos t + C$$

   To determine the value of C, we note that $s(0) = 3\cos(0) + C = 3 + C = 4$, so $C = 1$. Therefore,

   $$\boxed{s(t) = (2/3)t^{3/2} + 3\cos t + 1}$$

   (b) The integrand, $f(x) = 1/x^2$ is not continuous on the interval $[-2, 1]$ so you can not construct a Riemann sum for this function and the Fundamental Theorem of Calculus, Part 2 (or the Net Evaluation Theorem) does not apply.

   (c) We need a u-substititon. Let $u = ax^3 + 9bx + 1$. Then, $du = (3ax^2 + 9b)\, dx$. We also will change the limits of integration: when $x = 0$, $u = 1$ and when $x = 2$, $u = 8a + 18b + 1$. Thus,

   $$\begin{aligned}\int_0^2 \frac{ax^2+3b}{(ax^3+9bx+1)^{1/3}}\, dx &= \int_1^{8a+18b+1} \frac{1}{3u^{1/3}}\, du \\ &= \frac{3}{3\cdot 2} u^{2/3} \Big|_1^{8a+18b+1} \\ &= \boxed{\frac{1}{2}(8a+18b+1)^{2/3} - \frac{1}{2}}\end{aligned}$$

2. (17 points)

   (a) Find the value of the sum, $\displaystyle\sum_{i=1}^{n} \left(1 + \frac{3i}{n}\right)^2 \frac{3}{n}$, in terms of n.

   (b) Evaluate $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{3i}{n}\right)^2 \frac{3}{n}$

   (c) What is the definite integral that corresponds to the quantity in part (b)?

Solution:

(a)

$$\begin{aligned}
\sum_{i=1}^{n}\left(1+\frac{3i}{n}\right)^2\frac{3}{n} &= \sum_{i=1}^{n}\left(1+\frac{6i}{n}+\frac{9i^2}{n^2}\right)\frac{3}{n}\\
&= \frac{3}{n}\sum_{i=1}^{n}1+\frac{18}{n^2}\sum_{i=1}^{n}i+\frac{27}{n^2}\sum_{i=1}^{n}i^2\\
&= \frac{3}{n}\cdot n+\frac{18}{n^2}\frac{n(n+1)}{2}+\frac{27}{n^3}\frac{n(n+1)(2n+1)}{6}\\
&= \boxed{21+\frac{45}{2n}+\frac{9}{2n^2}}
\end{aligned}$$

(b)

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{3i}{n}\right)^2\frac{3}{n}=\lim_{n\to\infty}\left(21+\frac{45}{n}+\frac{9}{2n^2}\right)=\boxed{21}$$

(c)

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{3i}{n}\right)^2\frac{3}{n}=\boxed{\int_0^3(1+x)^2\,dx}$$

Alternate solution:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{3i}{n}\right)^2\frac{3}{n}=\boxed{\int_1^4 x^2\,dx}$$

Both integrals lead to the same Riemann sum and the same value of 21!

3. (21 points) Unrelated, short-answer questions:

   (a) Without trying to compute the integral, provide a reasonable upper bound and a reasonable lower bound for $\displaystyle\int_{-2}^{-1}(1+t^4)^{1/2}\,dt$. Briefly explain your reasoning.

   (b) Find the slant asymptote for $g(x)=\dfrac{x^3}{x^2+x-9}$.

   (c) Estimate $(17)^{1/4}$ by using Newton's method on $f(x)=x^4-17$ as follows: Do one iteration with $x_0=2$. What is x_1?

Solution:

(a) We know from the Comparison Properties of the Integral that if $m\le f(x)\le M$ for $a\le x\le b$, then

$$m(b-a)\le\int_a^b f(x)\,dx\le M(b-a)$$

For $f(t)=(1+t^4)^{1/2}$ and $-2\le t\le -1$ we have

$$f(-1)=(1+(-1)^4)^{1/2}=2^{1/2}\le(1+t^4)^{1/2}\le(1+(-2)^4)^{1/2}=(17)^{1/2}=f(-2)$$

And, since $b-a=-1-(-2)=1$, we have

$$\boxed{2^{1/2}\le\int_{-2}^{-1}(1+t^4)^{1/2}\,dt\le(17)^{1/2}}$$

(b) Long division yields $g(x) = \dfrac{x^3}{x^2+x-9} = x - 1 + \dfrac{10x-9}{x^2+x-9}$ so the slant asymptote is $\boxed{\text{y = x-1}}$.

(c) Apply Newton's method to $f(x) = x^4 - 17$ to obtain

$$x_{n+1} = x_n - \frac{f(x-n}{f'(x_n)} = x_n - \frac{x^4-17}{4x^3}$$

If $x_0 = 2$ we have $x_1 = 2 - \dfrac{16-17}{4\cdot 2^3} = \boxed{2+1/32} = \boxed{\dfrac{65}{32}}$

4. (16 points) You plan to make a box with a square base. The volume of the box will be 20 cubic inches. The material for the bottom of the box costs \$8 per square inch. The material for the sides of the box cost \$4 per square inch. The material for the top of the box costs \$12 per square inch. What are the dimensions of this box that will minimize the cost of the materials? **Solution:**

Let x be the length of one side of the bottom of the box and let y be the height. We are given that the volume $V = 20\text{in}^3 = x^2y$. We want to minimize the cost. So, we need to find an equation for the cost, C:

$$\begin{aligned} C &= \text{cost of bottom} + 4\cdot\text{cost of one side} + \text{cost of top} \\ &= 8x^2 + 4\cdot 4xy + 12x^2 \\ &= 20x^2 + 16xy \end{aligned}$$

We are given that $20 = x^2y$ so we can substitute $y = 20/x^2$ into the equation for cost. Thus, we obtain

$$C(x) = 20x^2 + \frac{16\cdot 20}{x}$$

The domain for the cost function is all $x > 0$

$$C'(x) = 40x - \frac{16\cdot 20}{x^2}$$

Set $C'(x) = 0$ to find the critical points: $40x = \dfrac{16\cdot 20}{x^2}$. This yields $x^3 = 8$ so $x = 2$. So, $x = 2$ is the only critical point on the domain $x > 0$. To verify that this gives the minimum, we take the second derivative $C''(x) = 40 + \dfrac{640}{x^3}$. Since the second derivative is positive for all $x > 0$, the second derivative test tells us that $x = 2$ minimizes the cost. So,the dimensions of the box are $\boxed{\text{x=2 and y=5}}$.

5. (24 points) For this problem, consider the graph of the function $f(x)$, continuous for all x and shown below on the interval $[-1, 4]$. For parts (a) - (d) find the requested information.

(a) Find the upper Riemann sum for f on the interval $[-1, 4]$ using 5 equally spaced subintervals.

(b) Find the average value of f on the interval $[-1, 4]$.

Now, let $p(x) = x + \int_0^{x^2} f(t)\, dt$ with f still given by the figure above.

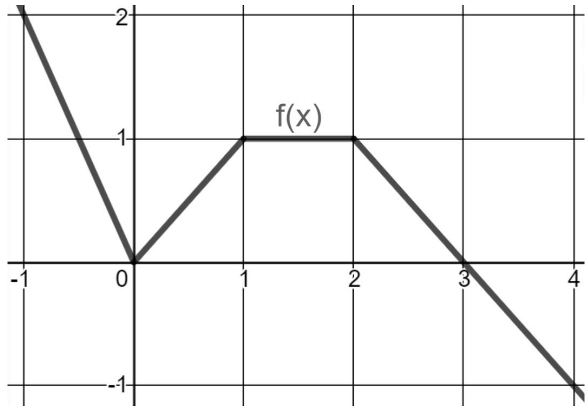
(c) Find $p(1)$.

(d) Find $p'(2)$.

Solution:

(a) On each of the five subintervals the maximum value of the function is 2, 1, 1, 1, 0, respectively. The width of each subinterval is 1, so the upper Riemann sum is $1(2 + 1 + 1 + 1 + 0) = 5$

(b) The average value formula is $\frac{1}{b-a}\int_a^b f(x)dx$. The integral of $f(x)$ on the interval is 2.5 and the length of the interval is 5, so the average value is $\frac{1}{2}$.



(c) $p(1) = 1 + \int_0^1 f(x)dx = 1.5$

(d) $p'(x) = 1 + 2xf(x^2)$, so $p'(2) = 1 + 4f(4) = -3$.

6. At the bottom of your work for problem 5: please write a statement that says your work is your own and you did not receive any help on the exam. Sign this statement. **Verify that everything has been uploaded correctly and pages have been associated to the correct problem before you leave Proctorio or the zoom proctoring room.**

Midterm 3 formulas:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$